

Comparison of two low-order basis functions for the volume integral equation

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1. Introduction

The MoM solution for inhomogeneous (and homogeneous) dielectric objects can be obtained using the method of volume integral equation [1]. The method of volume integral equation has the number of advantages including the applicability to inhomogeneous dielectric materials [2] as well as a potentially better accuracy at the resonances (compared to the surface integral formulation [3]). At the same time, it suffers from a rapid growth of computational complexity with increasing the size of the grid. Therefore, possible reduction of the number of basis functions (unknowns) and simplification of the basis functions themselves may improve the performance of the method.

The simplest choice for the volume integral equation is the pulse basis functions (cf. [4]). However, they tend to be unstable when relative permittivity becomes higher [1]. The face-based tetrahedral basis functions proposed by Schaubert, Wilton, and Glisson [2] are more robust and are increasingly used today [3, 5].

A convenient choice is to select the electric flux density $\vec{D}$ as the unknown for the volume integral equation [2]. Then, the continuity of the normal component of $\vec{D}$ must be enforced on the faces within the same basis function. This is contrast to the finite element method with the edge-based vector basis functions, where the continuity of the tangential $\vec{E}$ -field is required on the faces [6-8].

The face-based vector functions for the volume integral equation [2] are built from two tetrahedra sharing the same face; with a linear $\vec{D}$ variation essentially in the direction of the face normal. These basis functions exactly replicate any constant $\vec{D}$ -flux, but not its linear variation. The number of unknowns is equal to the number of the faces of the tetrahedral mesh. Note that, for a non-trivial tetrahedral mesh, the number of faces is larger than the number of the edges. This is a disadvantage compared to the edge-based functions, where the number of *independent* unknowns for the system matrix may be even smaller than the number of the edges [9, 10].

It is therefore inviting to employ some “mixed” basis functions that still acquire the condition of the continuous normal $\vec{D}$ - component according to [2], but include all

tetrahedra sharing the same edge – similar to the edge-based divergence-free basis functions [7]. These functions may be even piecewise constant.

The present paper compares the relative performance of one such set of basis functions versus the well-established face-based functions [2]. It is organized as follows. Section 2 gives the description of the mixed basis functions and estimates the necessary number of unknowns. Section 3 studies three examples: scattering of a homogeneous dielectric plate at normal incidence, scattering of an inhomogeneous finite cylinder at normal incidence, and an isolated dielectric resonator with medium-to-low Q . Section 4 concludes the paper.

2. Mixed basis functions

2.1. Definition¹

A mixed basis function $\vec{f}$ is shown in Fig. 1. It is introduced in a way that is somewhat similar to the first Whitney form [7]. However, the vector field variation is essentially perpendicular to the base edge $\vec{l}$ (or AB) whereas for the edge-based finite-element functions it is essentially parallel to it [7]. The basis function is defined through a vector of the edge $\vec{p}$ (or CD), which is opposite to the base edge $\vec{l}$; its direction follows the right-hand rule. Everywhere within a single tetrahedron, the basis function is a constant vector field given by

$$\vec{f} = c\vec{p} \tag{1}$$

where c is a normalization coefficient, which will be defined later on.

The basis function may include a different number of tetrahedra that share the same base edge $\vec{l}$. Three representative cases are depicted in Fig. 1. In the first case (Fig. 1a), both grayed faces of a single tetrahedron are on the mesh boundary. The basis function thus includes only one tetrahedron. In the second case (Fig. 1b), two faces of two adjacent tetrahedra are on the mesh boundary. The basis function includes two tetrahedra and has one inner face. In the last case (Fig. 1c), all faces of all tetrahedra

¹ The idea of the basis function shown in Fig. 1 belongs to the Ph.D. student in the ECE Department at Worcester Polytechnic Institute (WPI) Rosti Lemdiasov.

sharing the base edge are the inner faces of the mesh. The basis function includes all these tetrahedra and only has the inner faces. Thus, for any basis function, the number of boundary faces is either two or zero.

The component of the basis function $\vec{f}$ normal to face ABC in Fig. 1a is given by

$$f_{\perp} = c\vec{p} \cdot \vec{n}_{ABC} = c\vec{p}_{\perp} \cdot \vec{n}_{ABC} = c \frac{2\bar{S}}{h_C} = c \frac{2\bar{S}}{2S_{ABC}/l} = c \frac{l\bar{S}}{S_{ABC}} \quad (2)$$

where $\vec{p}_{\perp}$ is the projection of $\vec{p}$ onto a plane perpendicular to the base edge; h_C is the height of triangle ABC perpendicular to the base edge; and $\bar{S}$ is the area of the projection of triangle ACD or triangle BCD onto a plane perpendicular to the base edge. The projection triangle is shown by a dashed line in Fig. 1a. Note that $\vec{p}_{\perp} \cdot \vec{n}_{ABC}$ is the height of the projection triangle, which is perpendicular to its side h_C . The normalization coefficient is chosen in the form

$$c = 1/l\bar{S} \quad (3)$$

This guarantees that (i) the normal component of the basis function is continuous through the inner faces; and (ii) the total flux of the normal component through any face is equal to one.

2.2. The size of the function set

A naive guess is to assume that the number of basis functions is equal to the number of edges N_L of the tetrahedral mesh. This approach leads to the ill-conditioned Fourier expansion matrix. In order to estimate the number of independent basis functions, let us first consider a mesh with one tetrahedron shown in Fig. 2a. Formally, there are six basis functions corresponding to six basis edges. Only three of them are linearly independent in $\mathfrak{R}^3$, say, $\vec{l}_1, \vec{l}_2$, and $\vec{l}_3$, and should therefore be retained. The number of basis function is thus

$$N = N_F - N_T \quad (4)$$

where N_F is the number of faces (four) and N_T is the number of tetrahedra (one) in the mesh.

Next, let us consider a mesh with two tetrahedra shown in Fig 2b. The component of the field, which is perpendicular to the common face, is the same in both tetrahedra and is supported by basis function $\vec{l}_1$. The remaining component of the field, which is parallel to the face, is different in both tetrahedra, and is supported by two basis functions, say $\vec{l}_2, \vec{l}_3$, and $\vec{l}_4, \vec{l}_5$, respectively. The number of independent basis functions (five) is again given by equation (4), where $N_F = 7, N_T = 2$.

In order to justify equation (4) in a general case the following can be mentioned. For any tetrahedral mesh, only one basis function is needed per face to support the normal field component through the given face. This leads to the first term on the right-hand side of equation (4). On the other hand, any tetrahedron has four faces but needs only three linearly independent basis functions. Therefore, one basis function per tetrahedron must be subtracted. This leads to the second term on the right-hand side of equation (4). Equation (4) is validated directly for a number of different meshes².

To identify the linearly-independent basis functions for a given tetrahedral mesh, the following approach is chosen. The expansion or ‘‘covariance’’ matrix of a set of basis functions on the size $N_L \times N_L$ is set in the form

$$C = \begin{bmatrix} \langle \vec{f}_1, \vec{f}_1 \rangle & \dots & \langle \vec{f}_1, \vec{f}_{N_L} \rangle \\ \dots & \dots & \dots \\ \langle \vec{f}_{N_L}, \vec{f}_1 \rangle & \dots & \langle \vec{f}_{N_L}, \vec{f}_{N_L} \rangle \end{bmatrix}, \quad \langle \vec{f}_m, \vec{f}_n \rangle = \int_V \vec{f}_m \cdot \vec{f}_n d\vec{r} \quad (5)$$

where N_L is the number of edges in the mesh. The independent columns of matrix C correspond to independent basis functions. They can be found in a number of ways. One of them is to reduce matrix C by row operations to an echelon form, E , using Gauss-Jordan elimination [11]³. Then, the basic columns of matrix E are exactly in the same

² The reasoning given is not the exact proof of equation (4). Another result is that the number of independent unknowns is three times the number of tetrahedra minus the number of inner faces.

³ The LAPACK function `dsytrf` can be used for symmetric real matrixes. The above operation is also implemented in Matlab® as function `rref`.

position as the linearly-independent columns of C [11]. Only these columns are retained. The nullspace of matrix C is thus eliminated from consideration⁴.

Even though matrix C has a greater size than the resulting impedance matrix, it is still real and symmetric. That reduces the memory usage and the computation speed compared to the operations on the impedance matrix.

2.3. Field and charge expansion

The total electric flux, $\vec{D}(\vec{r})$, has a continuous normal component and is expanded in the form

$$\vec{D}(\vec{r}) = \sum_{n=1}^N D_n \vec{f}_n(\vec{r}) \quad (6)$$

Once equation (6) is given, the density of the surface bound charges is established following the continuity equation, in terms of the surface δ -functions [1]. The equivalent result can be obtained using Gauss theorem or the boundary condition on the dielectric-dielectric interface. Consider two arbitrary tetrahedra (plus and minus) that share a common face (which includes the base edge) but have different dielectric constants $\hat{\epsilon}^\pm$. The surface charge density σ_s , from Gauss law,

$$\sigma_s \equiv (K_+ - K_-)D_\perp = \hat{K}D_\perp \quad (7)$$

where $D_\perp$ is the normal component of the total electric displacement on the boundary;

K_+, K_- are the corresponding contrasts. The dielectric contrast $K_\pm = \frac{\hat{\epsilon}^\pm - \epsilon_o}{\hat{\epsilon}^\pm}$ is a constant within every tetrahedron. The surface normal is directed from plus (or left) to minus (right) tetrahedron.

⁴ The situation described above is somewhat similar to the edge-based finite elements bases [9]. In the case of finite elements, the singular system matrix appears after the application of the *curl-curl* operator. In the present example, the Fourier expansion matrix itself is already singular. It is believed that the reason for the difference is a piecewise-constant behavior of the mixed basis functions, which is acceptable in an integral equation.

For every basis function $\vec{f}_n(\vec{r})$, and for every face, which supports the normal component of this basis function, the associated surface charge is enforced to follow equation (7). The normal component of $\vec{f}_n(\vec{r})$ is available from equation (2). The total surface charge density in the dielectric is obtained as a combination of the contributions of all basis functions

$$\sigma_s(\vec{r}) \equiv \hat{K}(\vec{r})D_\perp(\vec{r}) = \sum_{n=1}^N \left\{ \sum_{q=1}^Q \hat{K}_q f_{\perp nq}(\vec{r}) \right\} D_n \quad \vec{r} \in S \quad (8)$$

where $\hat{K}_q = K_+ - K_-$ is the differential contrast on face q , $f_{\perp nq}(\vec{r})$ is the normal component of the basis function $\vec{f}_n(\vec{r})$ on face q . The inner summation in equation (8) is done over all Q faces, which support the normal component of the n -th basis function.

The face normal is aligned according to the right-hand rule for the base edge. Its direction follows the direction of the vector field shown in Fig. 1. The differential contrast is obtained combining the contrast data for the left and right tetrahedron. If one of them does not exist (vacuum-dielectric boundary), then either K_+ or K_- becomes zero. For a homogeneous dielectric this guarantees that (i) every basis function with only inner faces doesn't create any surface charge; and (ii) every basis function with two boundary faces possesses the net surface charge zero.

Along with (8), the volume polarization current density in the dielectric volume, except for any of its boundaries, is given by

$$\vec{J}_v(\vec{r}) \equiv j\omega K(\vec{r})\vec{D}(\vec{r}) = j\omega \sum_{n=1}^N \left\{ \sum_{p=1}^P K_p \vec{f}_{np}(\vec{r}) \right\} D_n \quad \vec{r} \in V, \vec{r} \notin S \quad (9)$$

The inner summation in equation (9) is done over all P tetrahedra, which are contained by the n -th basis function. Every tetrahedron may possess its own dielectric contrast K_p .

2.4. MoM equations

According to the volume equivalence principle [1], the piecewise inhomogeneous dielectric material is removed and replaced by equivalent volume polarization currents in

V and by the associated surface bound charges on S . The volume EFIE is written in the mixed-potential form [2]

$$\vec{E}^i = \vec{E} + j\omega\vec{A}(\vec{r}) + \nabla\Phi(\vec{r}) \quad \vec{r} \in V, \vec{r} \in S \quad (10)$$

where $\vec{E} = \vec{D}/\hat{\epsilon}$ is the net electric field and $\vec{E}^i$ is the incident field. The magnetic vector potential $\vec{A}(\vec{r})$ describes radiation of volume polarization currents given by equation (9), whereas the electric potential $\Phi(\vec{r})$ describes radiation of the associated bound charges given by equation (8). One has

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \vec{J}_v(\vec{r}') g(\vec{r}, \vec{r}') d\vec{r}', \quad \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_S \sigma_s(\vec{r}') g(\vec{r}, \vec{r}') d\vec{r}' \quad (11)$$

where $g = \exp(-jkR)/R$, $R = |\vec{r} - \vec{r}'|$ is the free-space Green's function.

Multiplication of equation (10) by $K(\vec{r})\vec{f}_m(\vec{r})$, integration over dielectric volume V , and finally using Stokes theorem for every individual tetrahedron contained by $\vec{f}_m(\vec{r})$ gives N moment equations. The resulting surface integrals must be combined in such a way to extract terms related to differential contrasts $\hat{K}_q$. This gives

$$\begin{aligned} \sum_{p=1}^P K_p \int_{V_p} \vec{f}_{mp}(\vec{r}) \cdot \vec{E}^i d\vec{r} &= \sum_{p=1}^P \frac{K_p}{\hat{\epsilon}_p} \int_{V_p} \vec{f}_{mp}(\vec{r}) \cdot \vec{D}(\vec{r}) d\vec{r} + j\omega \sum_{p=1}^P K_p \int_{V_p} \vec{f}_{mp}(\vec{r}) \cdot \vec{A} d\vec{r} + \\ \sum_{q=1}^Q \hat{K}_q \int_{S_q} f_{\perp mq}(\vec{r}) \Phi d\vec{r} \quad m &= 1, \dots, N \end{aligned} \quad (12)$$

After substitution of (8), (9), and (11), equation (12) gives the MoM equations in the form

$$v_m = \sum_{n=1}^N Z_{mn} D_n \quad v_m = \sum_{p=1}^P K_p \int_{V_p} \vec{f}_{mp}(\vec{r}) \cdot \vec{E}^i(\vec{r}) d\vec{r} \quad (13)$$

where the impedance matrix Z is given by

$$\begin{aligned}
Z_{mn} = & \sum_{p=1}^P \sum_{p'=1}^{P'} \frac{K_p}{\hat{\epsilon}_p} \int_{V_p} \vec{f}_{mp}(\vec{r}) \cdot \vec{f}_{np'}(\vec{r}) d\vec{r} \\
& - \frac{\omega^2 \mu_0}{4\pi} \sum_{p=1}^P \sum_{p'=1}^{P'} K_p K_{p'} \int_{V_p} \int_{V_{p'}} g(\vec{r}, \vec{r}') \vec{f}_{mp}(\vec{r}) \cdot \vec{f}_{np'}(\vec{r}') d\vec{r} d\vec{r}' \\
& + \frac{1}{4\pi\epsilon_0} \sum_{q=1}^Q \sum_{q'=1}^{Q'} \hat{K}_q \hat{K}_{q'} \int_{S_q} \int_{S_{q'}} g(\vec{r}, \vec{r}') f_{\perp mq}(\vec{r}) f_{\perp nq'}(\vec{r}') d\vec{r} d\vec{r}' \quad m, n = 1, \dots, N
\end{aligned} \tag{14}$$

The symmetric impedance matrix is thus written as a combination of individual volume and surface integrals. Since both the basis/test functions and their normal components are constant for a given tetrahedron/face, equation (14) may be notably simplified. In terms of the notations of subsection 2.1 one has

$$\begin{aligned}
Z_{mn} = & \sum_{p=1}^P \sum_{p'=1}^{P'} \frac{K_p V_p}{\hat{\epsilon}_p l_m l_n \bar{S}_{mp} \bar{S}_{np'}} (\vec{p}_{mp} \cdot \vec{p}_{np'}) \delta_{pp'} \\
& - \frac{\omega^2 \mu_0}{4\pi} \sum_{p=1}^P \sum_{p'=1}^{P'} \frac{K_p K_{p'}}{l_m l_n \bar{S}_{mp} \bar{S}_{np'}} (\vec{p}_{mp} \cdot \vec{p}_{np'}) \int_{V_p} \int_{V_{p'}} g(\vec{r}, \vec{r}') d\vec{r} d\vec{r}' \\
& + \frac{1}{4\pi\epsilon_0} \sum_{q=1}^Q \sum_{q'=1}^{Q'} \frac{\hat{K}_q \hat{K}_{q'}}{S_q S_{q'}} \int_{S_q} \int_{S_{q'}} g(\vec{r}, \vec{r}') d\vec{r} d\vec{r}' \quad m, n = 1, \dots, N
\end{aligned} \tag{15}$$

Note that the first term on the right-hand side of equation (15) is only different from zero when the p -th tetrahedron of basis function m coincides with the p' -th tetrahedron of basis function n .

3. Test examples

3.1. Implementation

Both sets of basis functions are implemented using the integrations formulas of the same degree of accuracy for test/source integrals. This allows us to obtain a symmetric impedance matrix. The 7-point Gaussian quadrature of the 5th degree of accuracy is used for the surface integrals [12]. The volume integrals are calculated using the 5-point Gaussian quadrature of the 3rd degree of accuracy [13].

The singular portions of the surface-to-surface self-integrals are found analytically, using the closed-form solution given in [14]. The singularity extraction is

also done for the volume self-integrals. The singular portions of the volume-to-volume self-integrals are first integrated analytically, using Gauss theorem [15]. The remaining integrals are non-singular and are evaluated numerically, using the 7-point Gaussian quadrature for faces and the 8-point barycentric quadrature for volumes. Such combination is found to give a reasonable compromise between accuracy and speed⁵. The impedance matrix is filled using two nested loop pairs: one pair of loops is over tetrahedra, and another pair of loops is over the faces with no-zero differential contrasts. This allows us to avoid multiple calculations of the same integrals.

The scattering problem is considered, with the excitation given by an incident plane wave. After the MoM equations are solved, the scattered electric field within the dielectric (the “near” field) is found from equation (6) for both types of the basis functions. This near field is most sensitive to the numerical errors of the solver. The resulting near field is then compared to the similar data obtained using Ansoft HFSS v8.5. The Ansoft solutions are obtained using fine tetrahedral meshes, with typically more than 15,000 tetrahedra per structure.

3.2. Example 1 – homogeneous dielectric plate

The first example is a homogeneous dielectric plate of the size 4x4x0.4 cm shown in Fig. 3a-d. The relative dielectric constant is 10. The frequency of the y-polarized plane wave incident in the negative z -direction is either 750 MHz (the plate size is $\approx \lambda_{\text{diel}}/3$) or 1,500 MHz (the plate size is $\approx 2\lambda_{\text{diel}}/3$). A tetrahedral mesh has one layer into the depth but variable discretization accuracy in the xy -plane (Fig. 3a-d). The resulting scattered electric field in the dielectric has the dominant y -component, which is compared to the corresponding Ansoft solution. The field is evaluated on a line $x = -0.1$ cm, -2 cm $< y < 2$ cm, $z = 0$ cm, which is slightly off the y -axis. The offset is introduced in order to avoid possible field calculations exactly along tetrahedral faces.

Fig. 4 shows the scattered field as a function of the distance along the y -axis for three meshes with 96, 384, and 864 tetrahedra at 750 MHz. The left column is for the mixed basis functions; the right column corresponds to the face-based functions. The

⁵ The error in the self-integrals was observed below 15%. Compared to the 8-point barycentric quadrature, the 5-point Gaussian volume quadrature gave a higher error for the volume-to volume self-integrals.

Ansoft solution involves about 64,000 tetrahedra and is given by a smooth dashed curve. The MoM solution is always stair like. The L_2 error E between two solutions is given by

$$E = \sqrt{\int_{-0.02}^{0.02} |E_y^{MoM} - E_y^{ANSOFT}|^2 dy / \int_{-0.02}^{0.02} |E_y^{ANSOFT}|^2 dy} \quad (16)$$

It is seen from Fig. 4 that the mixed basis functions possess nearly the same or a smaller error for the given mesh compared to the face-based functions. The number of unknowns is however, considerably smaller. For first three meshes shown in Fig. 3a-c, the number of mixed functions follows equation (4) and equals 144, 544, and 1,200. The number of face-based functions is approximately 1.7 times higher: 240, 928, and 2,064.

Fig. 5a, b shows the error E for both sets of basis functions for total of 14 meshes, ranging from 54 tetrahedra (3x3 plate) to 1,536 tetrahedra (16x16 plate). Fig. 5a corresponds to 750 MHz; Fig. 5b corresponds to 1,500 MHz, respectively. The error for the mixed functions is shown by a bold curve (crosses); the error for the face-based functions is shown by a thin curve (circles). With the current version of the solver, we were unable to obtain the solution for the face-based functions in the case of the finest mesh (16x16) due to memory limitations. The value of the error reduces in nearly the same proportion for both sets of basis functions. The mixed functions behave slightly better for the lower frequency. It should be noted that, for certain meshes, the face-based basis functions may create unexpectedly large oscillations of the solution.

The number of unknowns for all meshes used in this example is indicated in Fig. 5c. Bold line indicates mixed functions; thin line is related to the face-based functions. The number of face-based functions is 1.64 times higher for the smallest 3x3 mesh and 1.73 times higher for the finest 16x16 mesh.

3.3. Example 2 – finite inhomogeneous cylinder

The second example is an inhomogeneous finite cylinder of radius 2 cm and the height of 4 cm (Fig. 3e-h). The inhomogeneity is imposed in the axial direction. The multilayered cylinder has 4 horizontal layers, each with different dielectric constant (Fig. 3e). The height of every layer is 1 cm. The relative dielectric constant varies from 10 for the top section to 2.5 for the bottom section, in steps of 2.5. The frequency of the z -

polarized plane wave incident in the positive x -direction is fixed at 750 MHz. A tetrahedral mesh has variable discretization accuracy depending on the number of layers into the depth along the z -axis. The discretization accuracy of the cylinder cap also changes from a rough mesh in Fig. 3e, f to a more accurate representation in Fig. 3g, h. The resulting scattered field in the dielectric has the dominant z -component, which is compared to the corresponding Ansoft HFSS solution. The field is evaluated on the cylinder axis $x = 0$ cm, $y = 0$ cm, -2 cm $< z < 2$ cm.

Fig. 6a-d shows the scattered electric field as a function of the distance along the cylinder axis for four different cylinders depicted in Fig. 3e-h with 480, 960, 1,776 and 2,664 tetrahedra (1,064, 2,088, 3,820, and 5,698 faces). The corresponding Ansoft HFSS solution obtained for a mesh with about 30,000 tetrahedra is shown by a dashed curve. The MoM solution with the mixed basis functions (solid line) is always stair like. The L_2 error E between two solutions is calculated similar to equation (16). The minimum error is 4.0% for the finest mesh in Fig. 6d.

Fig. 6e shows the error E for both sets of basis functions for four different cylinder meshes. The bold line corresponds to the mixed basis functions. Thin line indicates the error of the face-based basis functions. The error reduces in both cases. However, the error for mixed elements is smaller. With the current version of the solver, we were unable to obtain the solution for the face-based functions in the case of the finest cylindrical mesh due to memory limitations.

The number of unknowns for four different meshes used in this example is shown in Fig. 6f. Bold line indicates mixed functions; dashed line is related to face-based functions. Note that the number of unknowns for the face-based functions is 1.82 times higher for the cylinder shown in Fig. 3e, 1.85 times higher for the cylinder in Fig. 3f, 1.87 times higher for the cylinder in Fig. 3g, and 1.88 times higher for the cylinder in Fig. 3h.

3.4. Example 3 – homogeneous rectangular dielectric resonator

The next example is an isolated rectangular-plate dielectric resonator. Note that in Ref. [3], the face-based basis functions were employed to find the resonant frequency and the Q -factor of the lowest $TE_{11\delta}$ resonant mode (cf., for example [16, 17]) for a

rectangular dielectric plate. In order to check another independent realization of the face-based functions, we adopt the data of Ref. [3] related to those basis functions directly.

The plate has the size of $14.98 \times 14.98 \times 7.48$ mm and is thin in the z -direction. The relative dielectric constant (lossy dielectric) is $\epsilon_r = 200 - j20$, $\epsilon_r = 200 - j2$, and $\epsilon_r = 200 - j0$, respectively. Since the dielectric constant (reflection coefficient) is high, the quality factor of the resonator is mostly determined by the losses in the dielectric at $\epsilon_r = 200 - j20$. However, the radiation losses may be noticeable when the loss tangent decreases. Note that the MoM method includes radiation losses into account.

The Ansoft HFSS eigenmode solution is obtained with a cubical PML enclosure, whose size is $100 \times 100 \times 100$ mm, optimized close to the first resonant frequency. A series of adaptive passes is used until Δf reaches 0.01% for $\epsilon_r = 200 - j20$ and $\epsilon_r = 200 - j2$. For zero dielectric loss, the accuracy of 0.015% has been achieved.

The eigenmode solution for the mixed functions is obtained by the search for local minima of the cost function of two variables

$$F(f, f') = |\det(Z(f + jf'))| \quad (17)$$

where f is the real part of frequency and $f' > 0$ is the imaginary part. The search procedure is similar to that described in [18]. The resonant frequency and the quality factor of the resonator are then obtained as

$$f_{res} = f_{\min}, Q = \frac{f_{\min}}{2f'_{\min}} \quad (18)$$

Since the search procedure is time-consuming for larger meshes, the accuracy is limited by $\pm 0.2\%$ for both the real part of resonant frequency f and imaginary part f' .

Table 1 shows the data on f_{res} and Q found using three different approaches: Ansoft HFSS solution, the face-based functions [3], and the mixed functions. The mesh quality is varied. Generally, the face-based functions appear to perform much better for a rough $4 \times 4 \times 1$ grid, at all values of the dielectric loss. For the zero dielectric loss, their performance degrades when the mesh is refined.

At the same time, the performance of the mixed basis functions improves with refining the mesh. The convergence history is close to that observed for the finite element solution. The error in the resonant frequency falls below 2% when the mesh contains more than 400 tetrahedra. A larger error in the Q -factor is expected in the last case, due to a high Q value. Note that, for the face-based functions, the number of unknowns would be 1.67 times higher for the 4x4x1 grid, 1.75 times higher for the 4x4x2 grid, and 1.78 times higher for the 6x6x2 grid.

4. Conclusions

A comparison was made of the two types of basis functions for the volume integral equation. Both functions impose the continuity of the normal component of the electric flux through the faces. The one basis function is that of Schaubert, Wilton, and Glisson and is face-based. Another basis function (the mixed function) to some extent adopts the idea of the edge-based vector basis functions from the finite element method and is also edge-based. The mixed basis functions include all tetrahedra sharing the same edge. In its present version, the mixed basis function is constant within every single tetrahedron.

Three given examples (homogeneous scatterer, inhomogeneous scatterer, and a dielectric resonator) demonstrate similar performance of two types of the basis functions for equal tetrahedral meshes. However, for the same tetrahedral mesh, the number of unknowns for the mixed basis functions is considerably smaller. The ratio of unknowns (face-based vs. mixed) ranges from 1.6 for rough plate meshes to 1.9 for some large meshes. The mixed basis functions are easily implemented into the MoM method. Their disadvantage is a preliminary condition “operation”, which implies the elimination of the nullspace of the basis set.

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Table 1. Resonant frequency f_{res} and quality factor Q for the lowest $TE_{11\delta}$ mode in the isolated dielectric resonator on the size 14.98×14.98×7.48 mm. For the face-based functions, the data of Ref. [3] is used.

ϵ_r	Ansoft HFSS f_{res} [GHz], Q	Structure mesh	Mixed basis functions (present study) $\frac{\Delta f_{res}}{f_{res}}, \frac{\Delta Q}{Q}, \%$	Face-based functions (Ref. [3]) $\frac{\Delta f_{res}}{f_{res}}, \frac{\Delta Q}{Q}, \%$
200-j20	1.37544 GHz 9.9075	3x3x1 54 tet	+5.5 -0.3	NA NA
200-j20	1.37544 GHz 9.9075	4x4x1 96 tet	+4.6 +0.8	- 0.2 - 0.05
200-j20	1.37544 GHz 9.9075	4x4x2 192 tet	+3.0 +0.0	NA NA
200-j20	1.37544 GHz 9.9075	6x6x2 432 tet	+1.9 -0.3	NA NA
200-j2	1.38085 GHz 81.3224	3x3x1 54 tet	+5.4 -3.0	NA NA
200-j2	1.38085 GHz 81.3224	4x4x1 96 tet	+4.6 -3.5	- 0.3 +1.2
200-j2	1.38085 GHz 81.3224	4x4x2 192 tet	+3.0 -2.0	NA NA
200-j2	1.38085 GHz 81.3224	6x6x2 432 tet	+1.8 -0.2	NA NA
200-j0	1.38111 GHz 419.334	3x3x1 54 tet	+5.5 -17.3	NA NA
200-j0	1.38111 GHz 419.334	4x4x1 96 tet	+4.5 -17.0	-0.3 +4.6
200-j0	1.38111 GHz 419.334	4x4x2 192 tet	+3.0 -13.0	+1.6 -6.8
200-j0	1.38111 GHz 419.334	6x6x2 432 tet	+1.7 -7.6	NA NA

Figure captions

Fig. 1. Three possible configurations of the mixed basis function: a) two faces on the mesh boundary and no inner face; b) two faces on the mesh boundary and one (or more) inner face(s); c) only inner faces and no boundary faces.

Fig. 2. Set of linearly independent mixed functions for a) mesh with one tetrahedron; b) – mesh with two adjacent tetrahedra.

Fig. 3. Square dielectric plate (left column) and inhomogeneous cylindrical structure (right column) used at the simulations. Tetrahedral meshes are obtained using `delaunay3` of Matlab®.

Fig. 4. The dominant component of the scattered field $|E_y^s|$ within the dielectric plate on the y-axis compared to Ansoft solution (dashed curve) at 750 MHz. Three groups of results are related to three different meshes. The left column corresponds to the mixed functions whereas the right column corresponds to the face-based functions.

Fig. 5. a) L_2 error of the mixed set (bold curve) and of the face-based set (thin curve) as a function of the total number of tetrahedra in the mesh at 750 MHz; b) the same data for 1,500 MHz. The discrete points corresponding to the meshes with the same number of tetrahedra are marked by crosses and circles, respectively; c) total number of unknowns for two sets of basis functions: thick line – mixed functions; thin line – face-based functions.

Fig. 6. a) - d) Dominant component of the scattered field $|E_z^s|$ within the axially inhomogeneous cylinder (solid curve) compared to Ansoft solution (dashed curve). Four plots a)-d) are related to four different meshes shown in Fig. 3e-h; e) L_2 error of the mixed set (bold line) and of the face-based set (thin line) as a function of the total number of tetrahedra in the mesh at 750 MHz; f) number of unknowns for mixed and face-based functions as a function of the total number of tetrahedra in the mesh.

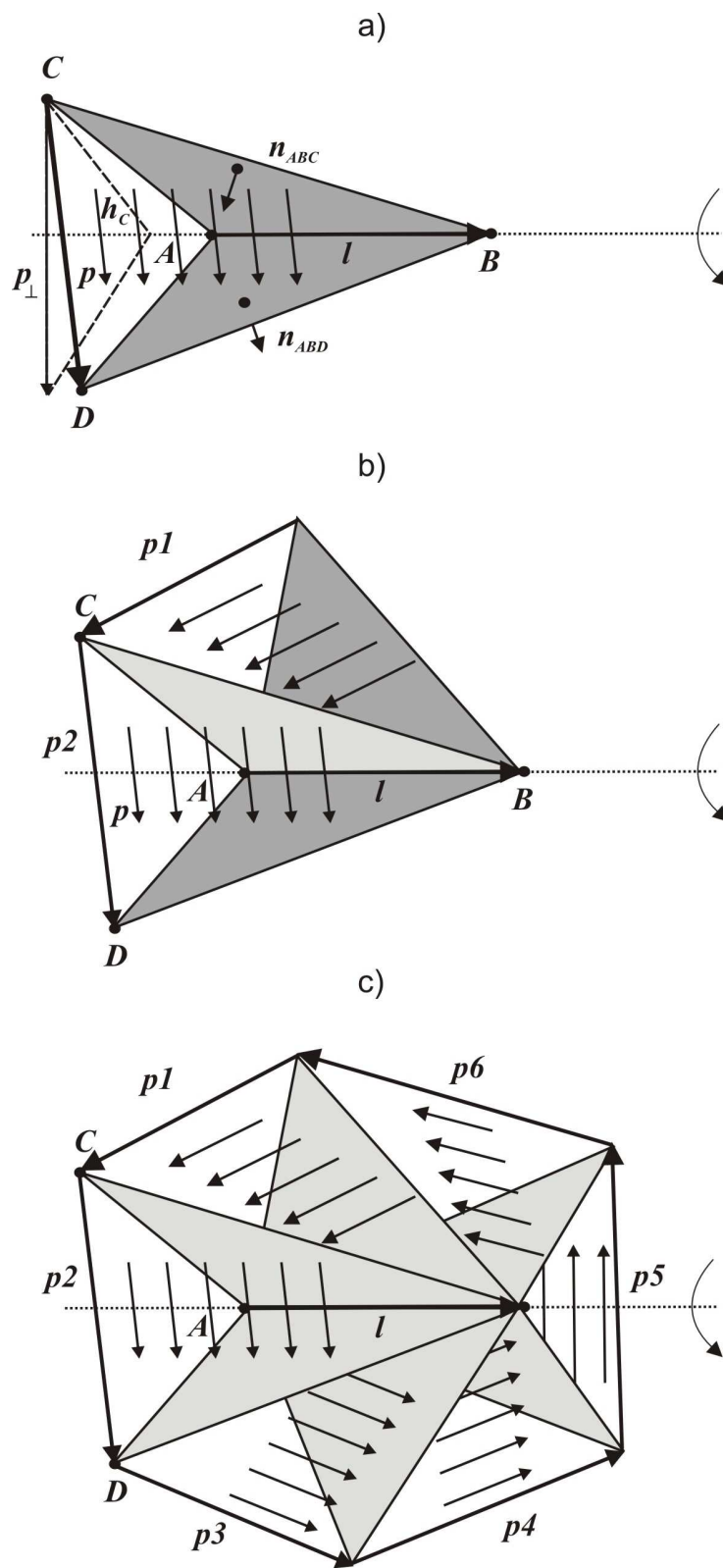


Fig. 1

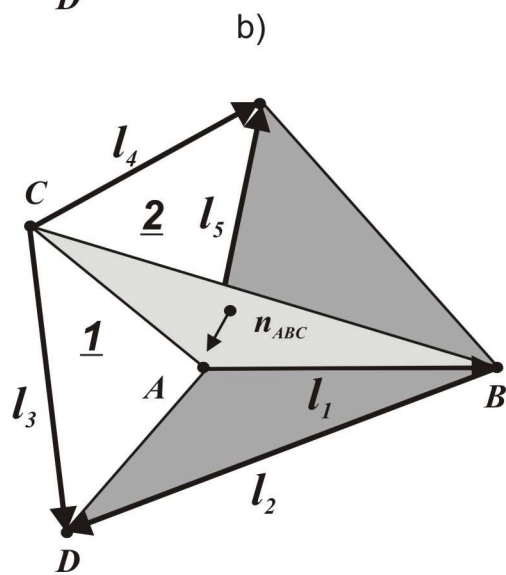
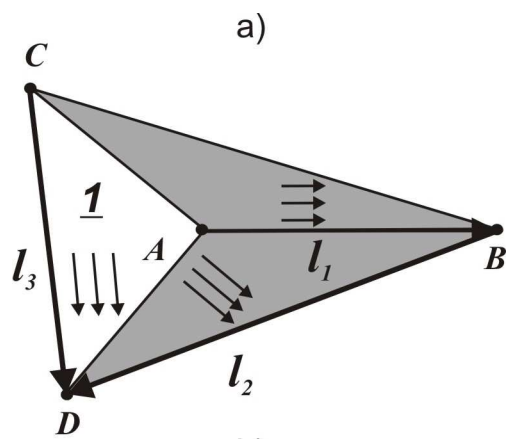


Fig. 2

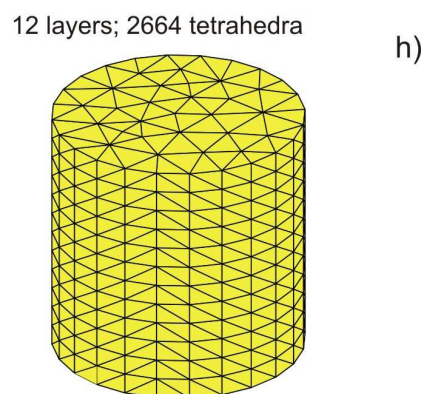
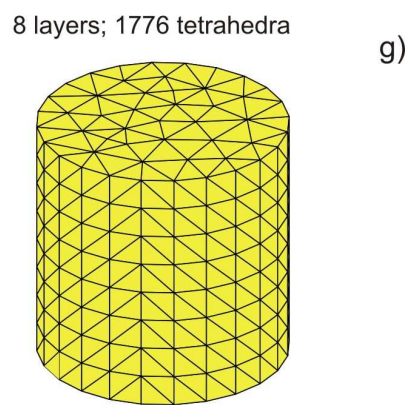
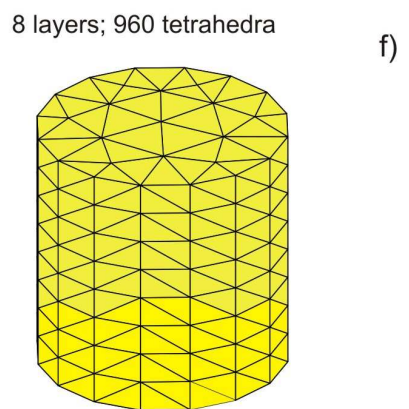
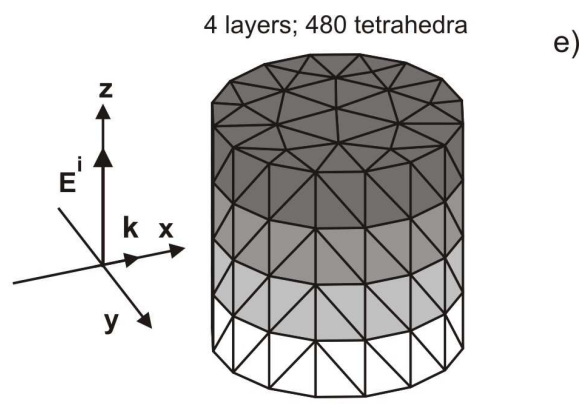
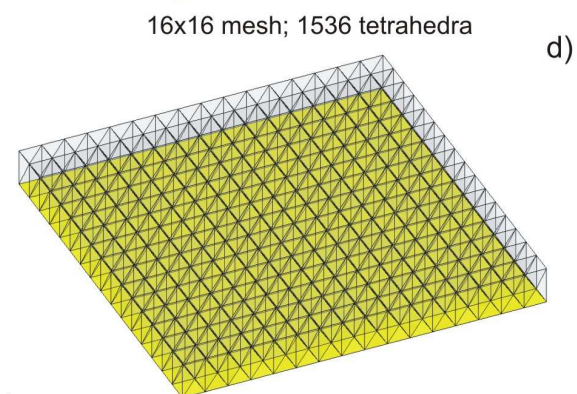
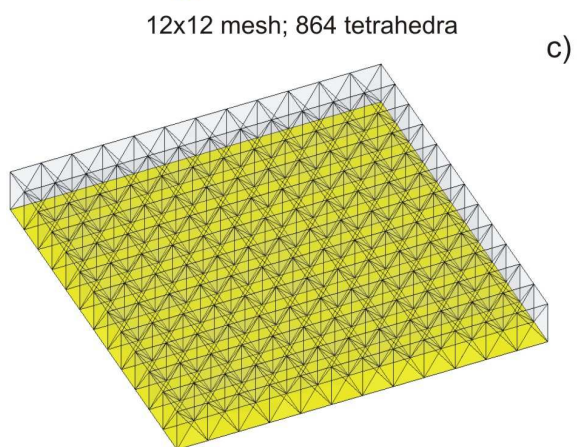
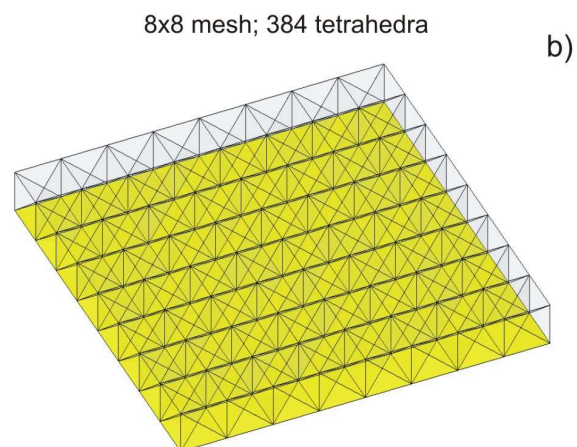
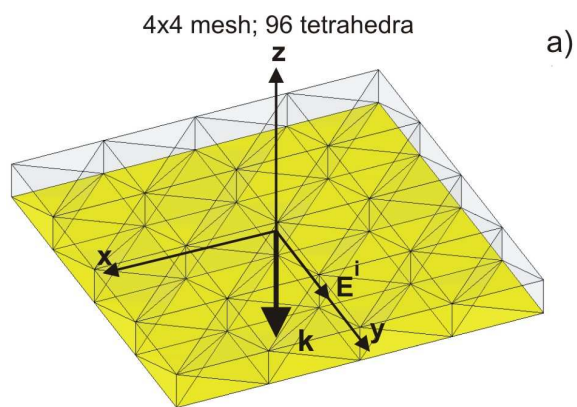
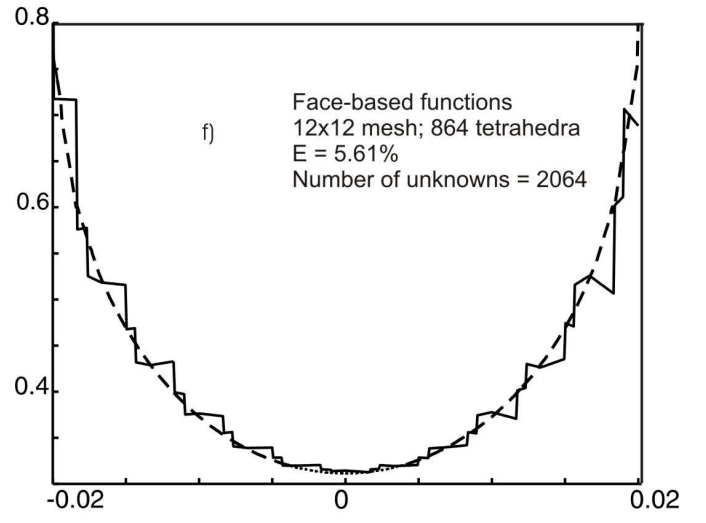
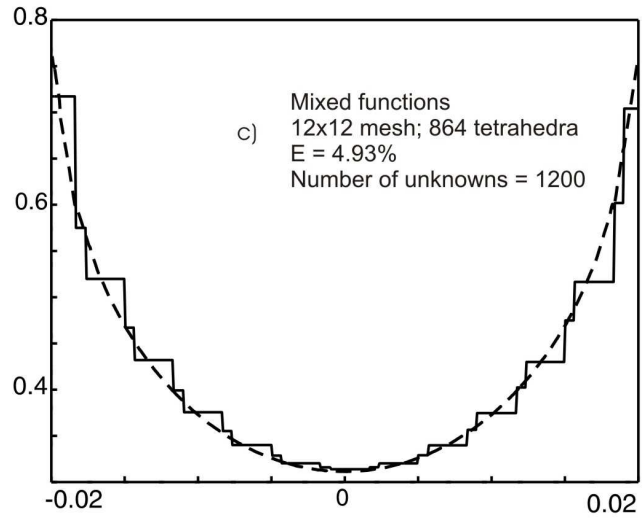
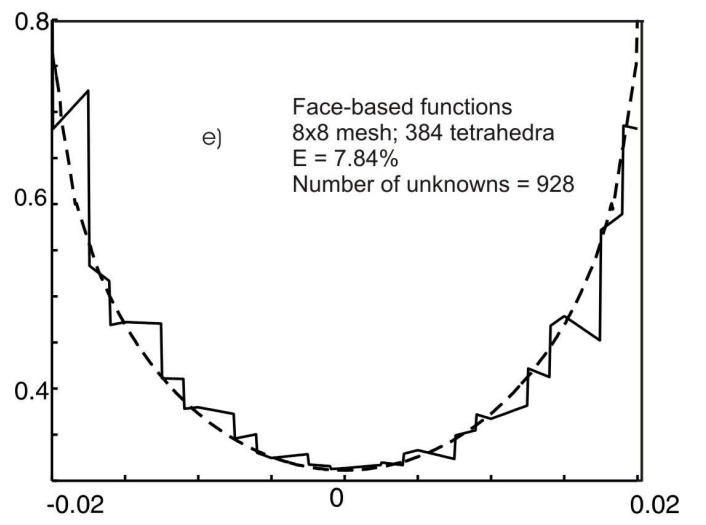
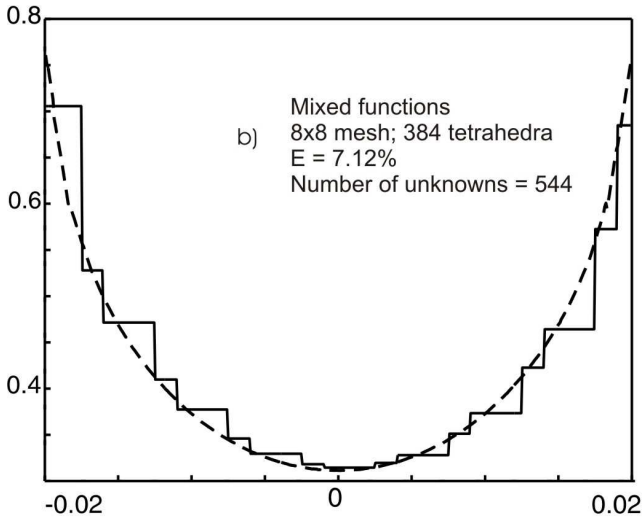
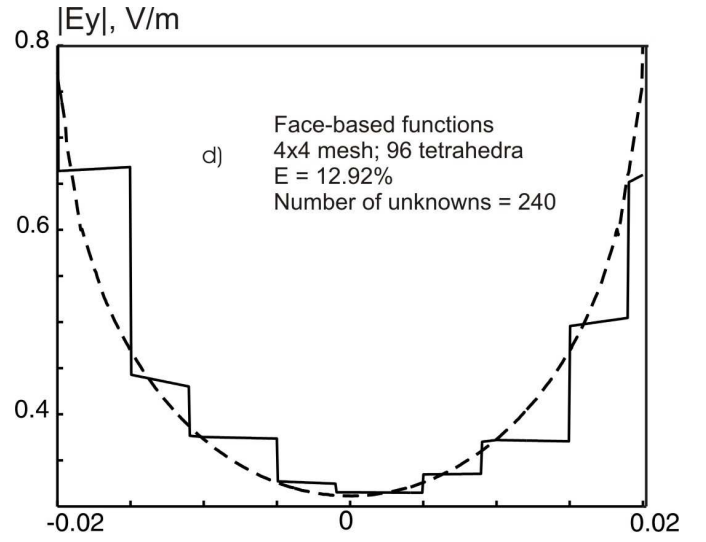
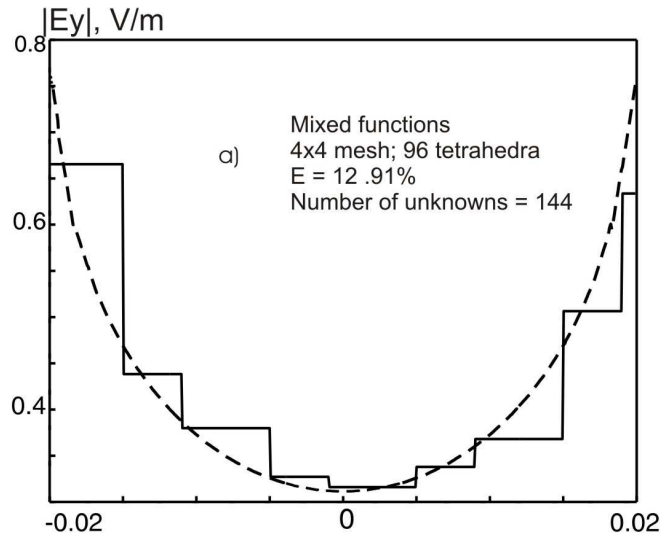


Fig. 3



Distance along y- axis , m

Distance along y- axis , m

Fig. 4

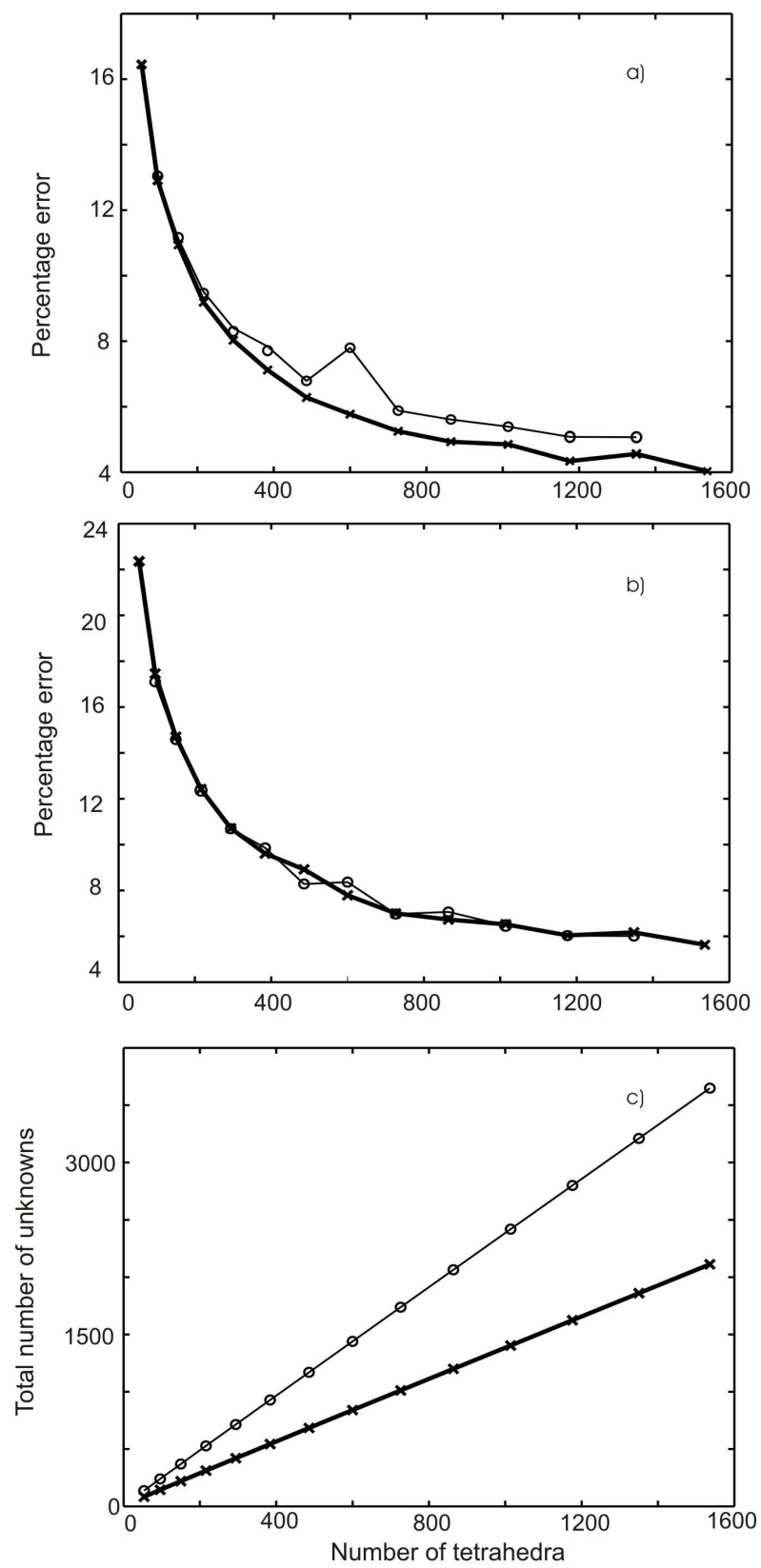


Fig. 5

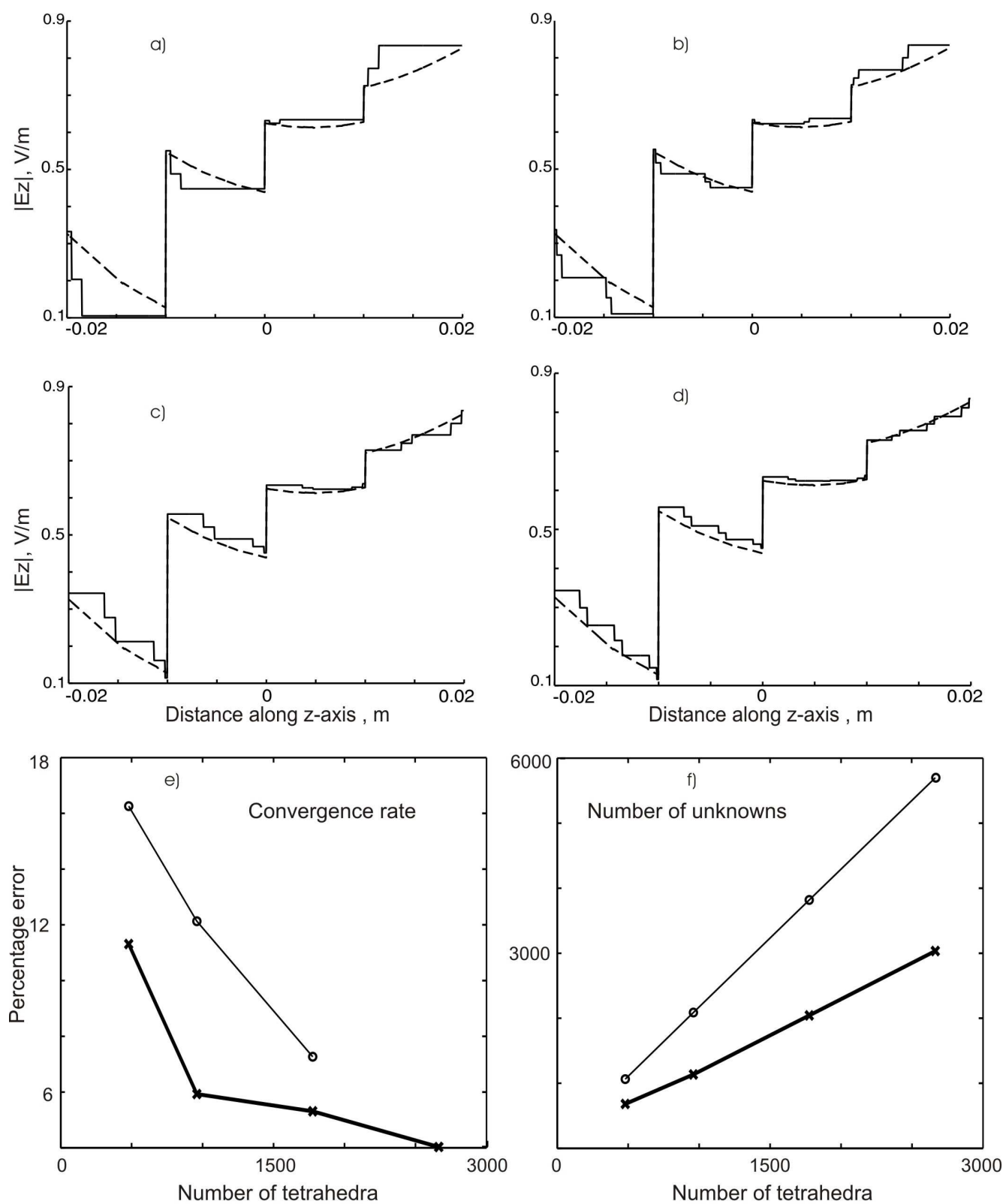


Fig. 6