

3. Theory

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Here, we recall the MoM equations and the near-field equations used in the present study. The electric field integral (EFIE) [1] and RWG basis functions [2] are employed, utilizing arbitrary integration accuracy and symmetry properties of the impedance matrix.

3.1. EFIE with symmetric impedance matrix

The total electric field is a combination of the incident field (labeled by superscript i) and the scattered field (labeled by superscript s), i.e.

$$\vec{E} = \vec{E}^i + \vec{E}^s \quad (1)$$

The scattered electric field $\vec{E}^s$ is due to surface currents and free charges on the metal surface S (the so-called mixed-potential formulation)

$$\vec{E}^s = -j\omega\vec{A}_s(\vec{r}) - \nabla\Phi_s(\vec{r}) \quad \vec{r} \text{ on } S \quad (2)$$

The magnetic vector potential $\vec{A}_s(\vec{r})$ describes current radiation whereas the electric potential $\Phi_s(\vec{r})$ describes charge radiation. In the near field, the Φ -contribution may be somewhat more critical than the $\vec{A}$ -contribution. In the far-field, the Φ -contribution is negligibly small. The mixed potential-expression (2) for the electric field is often preferred to other forms because it involves a less singular kernel, after evaluating in a weighted average sense on the metal surface S . On the metal surface S , $\vec{E}_{\text{tan}} = 0$, thus giving the EFIE [1],

$$E_{\text{tan}}^i = \left(j\omega\vec{A}_s(\vec{r}) + \nabla\Phi_s(\vec{r}) \right)_{\text{tan}} \quad \vec{r} \text{ on } S \quad (3)$$

Assume that some test functions, $\vec{f}_m^s(\vec{r})$ $m = 1, \dots, N$, cover the entire surface S and are zero everywhere except for a small subsurface $S_m \in S$. The test functions (i) do not have a component normal to the surface S_m and; (ii) do not have a component normal to the

surface border (if any). Multiplication of equation (3) by $\vec{f}_m^S$ and integration over S gives N equations

$$\int_{S_m} \vec{f}_m^S \cdot \vec{E}^i d\vec{r} = j\omega \int_{S_m} \vec{f}_m^S \cdot \vec{A}_S d\vec{r} - \int_{S_m} (\nabla \cdot \vec{f}_m^S) \Phi_S d\vec{r} \quad m = 1, \dots, N \quad (4)$$

since

$$\int_{S_m} \vec{f}_m^S \cdot (\vec{A}_S)_{\tan} d\vec{r} = \int_{S_m} \vec{f}_m^S \cdot \vec{A}_S d\vec{r}, \quad \int_{S_m} \vec{f}_m^S \cdot (\nabla \Phi_S)_{\tan} d\vec{r} = - \int_{S_m} \Phi_S (\nabla_S \cdot \vec{f}_m^S) d\vec{r} \quad (5)$$

Using the expansion for the surface current density

$$\vec{J}_S = \sum_{n=1}^N I_n \vec{f}_n^S(\vec{r}) \quad (6)$$

the magnetic vector potential due to metal surface currents becomes

$$\vec{A}_S(\vec{r}) = \frac{m_0}{4\pi} \int_S \vec{J}_S(\vec{r}') g(\vec{r}, \vec{r}') d\vec{r}' = \sum_{n=1}^N \left\{ \frac{m_0}{4\pi} \int_{S_n} \vec{f}_n^S(\vec{r}') g(\vec{r}, \vec{r}') d\vec{r}' \right\} I_n \quad (7)$$

where $g = \exp(-jkR)/R$, $R = |\vec{r} - \vec{r}'|$ is the free-space Green's function (time dependency $\exp(j\omega t)$ is assumed everywhere). Similarly, the electric potential due to free surface charge density $s_S = j/\omega \nabla_S \cdot \vec{J}_S$ takes the form

$$\Phi_S(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_S s_S(\vec{r}') g(\vec{r}, \vec{r}') d\vec{r}' = \sum_{n=1}^N \left\{ \frac{j}{4\pi\epsilon_0\omega} \int_{S_n} (\nabla_S \cdot \vec{f}_n^S(\vec{r}')) g(\vec{r}, \vec{r}') d\vec{r}' \right\} I_n, \quad (8)$$

The moment equations for the unknown “currents” I_n are obtained if we substitute expansions (7) and (8) into primary equation (4). In terms of symbolic notations,

$$u_m = \sum_{n=1}^N Z_{mn} I_n, \quad m = 1, \dots, N \quad (9)$$

where

$$u_m = \int_{S_m} \vec{f}_m^S \cdot \vec{E}^i d\vec{r} \quad (10)$$

are the “voltage” or excitation components for every test/basis function and

$$\begin{aligned}
Z_{mn} = & + \frac{j\omega\mu_0}{4p} \iint_{S_m S_n} \vec{f}_m^S(\vec{r}) \cdot \vec{f}_n^S(\vec{r}') g(\vec{r}, \vec{r}') d\vec{r}' d\vec{r} \\
& - \frac{j}{4p\omega\epsilon_0} \iint_{S_m S_n} (\nabla_s \cdot \vec{f}_m^S(\vec{r})) (\nabla_s \cdot \vec{f}_n^S(\vec{r}')) g(\vec{r}, \vec{r}') d\vec{r}' d\vec{r}
\end{aligned} \tag{11}$$

are the components of the impedance matrix of the size $N \times N$. Note that the impedance matrix is *symmetric* for any set of basis functions (test functions should be the same) when the corresponding surface integrals are calculated using the same degree of accuracy. The components of the impedance matrix are the double surface integrals of the Green's function and they mostly reflect the geometry of the problem.

3.2. Filling the symmetric impedance matrix with RWG basis functions

Below, we recall the following properties of the RWG basis functions [2]. For any two triangular patches, t_n^+ and t_n^- , having areas A_n^+ and A_n^- , and sharing the common edge l_n , the n -th basis function becomes

$$f_n^S(\vec{r}) = \begin{cases} \frac{l_n}{2A_n^+} \vec{r}_n^+ & \vec{r} \text{ in } t_n^+ \\ \frac{l_n}{2A_n^-} \vec{r}_n^- & \vec{r} \text{ in } t_n^- \end{cases} \quad \text{and} \quad \nabla_s \cdot f_n^S(\vec{r}) = \begin{cases} \frac{l_n}{A_n^+} & \vec{r} \text{ in } t_n^+ \\ -\frac{l_n}{A_n^-} & \vec{r} \text{ in } t_n^- \end{cases} \tag{12}$$

where $\vec{r}_n^+ = \vec{r} - \vec{r}_n^+$ is the vector drawn from free vertex of triangle t_n^+ to the observation point; $\vec{r}_n^- = \vec{r}_n^- - \vec{r}$ is the vector drawn from the observation point to the free vertex of triangle t_n^- . The n -th basis function is zero outside two adjacent triangles t_n^+ and t_n^- . Substitution of equations (12) into equation (11) gives the components of the impedance matrix in terms of the surface RWG basis functions in the form

$$\begin{aligned}
& \iint_{S_m S_n} \vec{f}_m^S \cdot \vec{f}_n^S g d\vec{r}' d\vec{r} = \\
& + \frac{l_m l_n}{4A_m^+ A_n^+} \iint_{t_m^+ t_n^+} (\vec{r}_m^+ \cdot \vec{r}_n^+) g d\vec{r}' d\vec{r} + \frac{l_m l_n}{4A_m^+ A_n^-} \iint_{t_m^+ t_n^-} (\vec{r}_m^+ \cdot \vec{r}_n^-) g d\vec{r}' d\vec{r} \\
& + \frac{l_m l_n}{4A_m^- A_n^+} \iint_{t_m^- t_n^+} (\vec{r}_m^- \cdot \vec{r}_n^+) g d\vec{r}' d\vec{r} + \frac{l_m l_n}{4A_m^- A_n^-} \iint_{t_m^- t_n^-} (\vec{r}_m^- \cdot \vec{r}_n^-) g d\vec{r}' d\vec{r}
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
& \int_{S_m} \int_{S_n} (\nabla_s \cdot \vec{f}_m^s) (\nabla_s \cdot \vec{f}_n^s) g d\vec{r}' d\vec{r} = \\
& + \frac{l_m l_n}{A_m^+ A_n^+} \int_{t_m^+ t_n^+} g d\vec{r}' d\vec{r} - \frac{l_m l_n}{A_m^+ A_n^-} \int_{t_m^+ t_n^-} g d\vec{r}' d\vec{r} \\
& - \frac{l_m l_n}{A_m^- A_n^+} \int_{t_m^- t_n^+} g d\vec{r}' d\vec{r} + \frac{l_m l_n}{A_m^- A_n^-} \int_{t_m^- t_n^-} g d\vec{r}' d\vec{r}
\end{aligned} \tag{14}$$

3.3. Accumulation of integrals into the impedance matrix

Evaluation of the MoM impedance matrix for RWG basis functions is by 90% calculation of the surface integrals presented in equations (13) and (14). Consider a structure where all triangular patches are enumerated by $p = 1, \dots, P$. Then, every integral in equation (13) is represented through a term

$$A_{pq}^{ij} = \int_{t_p} \int_{t_q} (\vec{r}_i \cdot \vec{r}_j') g(|\vec{r} - \vec{r}'|) d\vec{r}' d\vec{r} \quad p, q = 1, \dots, P \tag{15}$$

to within a constant factor. Here, $\vec{r}_i = \vec{r} - \vec{r}_i$ for any vertex i of patch p whereas $\vec{r}_j' = \vec{r}' - \vec{r}_j$ for any vertex j of patch q . Similarly, every integral in equation (14) is build based upon a term

$$\Phi_{pq} = \int_{t_p} \int_{t_q} g(|\vec{r} - \vec{r}'|) d\vec{r}' d\vec{r} \quad p, q = 1, \dots, P \tag{16}$$

All integrals (15) and (16) are first found for every given pair of patches p, q . Note that

$$\Phi_{pq} = \Phi_{qp}, \quad \bar{A}_{pq}^{ij} = \bar{A}_{qp}^{ji} \tag{17}$$

so that only the *half* of integrals needs to be calculated. Numerical integration of double surface integrals is done using the same Gaussian formula (of variable degree of accuracy) for both single surface integrals []. The most attractive formulas of third- and fourth degree of accuracy are given in Table A1.

Further, the contributions of every pair of patches p, q given by equations (15) and (16) are accumulated in the appropriate terms of equations (13) and (14), respectively. This happens when the basis functions m and n have p and q either as plus or minus triangles. Indexes i, j in equation (15) need to be chosen according to given m and n . The summation process for the impedance matrix has the following form:

$$Z_{mn} = Z_{mn} \pm \frac{l_m l_n}{A_p A_q} \left\{ \frac{j\omega m_0}{16p} A_{pq}^{ij} - \frac{j}{4p\omega e_0} \Phi_{pq} \right\} \tag{18}$$

Here, $m = m(p)$ are numbers of those RWG basic functions, which include patch p . Similarly, indexes $n = n(q)$ indicate basis functions, which include patch q . Plus sign occurs when p and q are both “plus” or “minus” triangles. Otherwise, sign minus should be chosen. Index $i = i(m)$ corresponds to the free vertex of patch p in element m . Index $j = j(n)$ corresponds to the free vertex of patch q in element n .

Table A1. Gaussian integration over triangular patches – third- and fourth degree of accuracy [3, 4].

Degree of precision = 3, number of points = 4		
Points	Triangular (vertex) coordinates	Weights
A	1/3, 1/3, 1/3	-27/48
B	0.6, 0.2, 0.2	25/48
C	0.2, 0.6, 0.2	25/48
D	0.2, 0.2, 0.6	25/48
Degree of precision = 4, number of points = 7		
Points	Triangular (vertex) coordinates	Weights
A	1/3, 1/3, 1/3	0.2250000000
B	a_1, β_1, β_1	0.1323941527
C	β_1, a_1, β_1	0.1323941527
D	β_1, β_1, a_1	0.1323941527
E	a_2, β_2, β_2	0.1259391805
F	β_2, a_2, β_2	0.1259391805
G	β_2, β_2, a_2	0.1259391805

with

$$a_1=0.0597158717, \beta_1=0.4701420641, a_2=0.7974269853, \beta_2=0.1012865073.$$

Note that the Gaussian quadrature only makes sense when the triangular patches are relative close to each other. Otherwise, the central-point approximation for both the source and test integrals is a good approximation to reality. The relative (to the triangle size) distance D between two triangular patches p and q may be defined in the following way:

$$D_{pq} = \sqrt{\frac{|\vec{r}_c^p - \vec{r}_c^q|^2}{\max_{i=1,2,3}(|\vec{r}_c^p - \vec{r}_i|) \max_{i=1,2,3}(|\vec{r}_c^q - \vec{r}_i|)}} \quad p, q = 1, \dots, P \quad (19)$$

where index c indicates patch centroid. The default criterion for performing Gaussian integration in the present study is

$$D_{pq} \leq 15 \quad (20)$$

If this inequality is not satisfied, the central-point approximation is used.

3.4. Self-integrals

The self-integrals ($p = q$ in equations (15), (16)) are found precisely, using a number of analytical base integrals presented in [5]. Before doing that, the Taylor expansion is made for the Green's function

$$g = \exp(-jkR)/R \approx 1/R - jk - k^2 R/2 \quad (21)$$

Therefore

$$\Phi_{pp} \approx \iint_{t_p t_p} \frac{1}{|\vec{r} - \vec{r}'|} d\vec{r}' d\vec{r} - jkA_p^2 - \frac{k^2}{2} \iint_{t_p t_p} |\vec{r} - \vec{r}'| d\vec{r}' d\vec{r} \quad (22)$$

and

$$\begin{aligned} A_{pp}^{ij} \approx & \iint_{t_p t_p} \frac{(\vec{r} - \vec{r}_i) \cdot (\vec{r}' - \vec{r}_j)}{|\vec{r} - \vec{r}'|} d\vec{r}' d\vec{r} - jk \iint_{t_p t_p} (\vec{r} - \vec{r}_i) \cdot (\vec{r}' - \vec{r}_j) d\vec{r}' d\vec{r} \\ & - \frac{k^2}{2} \iint_{t_p t_p} |\vec{r} - \vec{r}'| (\vec{r} - \vec{r}_i) \cdot (\vec{r}' - \vec{r}_j) d\vec{r}' d\vec{r} \end{aligned} \quad (23)$$

Only the first term on the right-hand side of equation (22) or (23) needs to be calculated analytically. The rest of the terms are found using Gaussian formulas discussed above, since they do not have a singular kernel. Introduction of triangular coordinates l_1, l_2 for the patch t_p gives [5]

$$\vec{r} = l_1(\vec{r}_1 - \vec{r}_3) + l_2(\vec{r}_2 - \vec{r}_3) + \vec{r}_3, \quad \vec{r}' = l'_1(\vec{r}_1 - \vec{r}_3) + l'_2(\vec{r}_2 - \vec{r}_3) + \vec{r}_3 \quad (24)$$

and

$$\begin{aligned} (\vec{r} - \vec{r}_i) \cdot (\vec{r}' - \vec{r}_j) = & (\vec{r}_3 - \vec{r}_i) \cdot (\vec{r}_3 - \vec{r}_j) \times 1 + \\ & (\vec{r}_1 - \vec{r}_3) \cdot (2\vec{r}_3 - \vec{r}_i - \vec{r}_j) \times l_1 + \\ & (\vec{r}_2 - \vec{r}_3) \cdot (2\vec{r}_3 - \vec{r}_i - \vec{r}_j) \times l_2 + \\ & (\vec{r}_1 - \vec{r}_3) \cdot (\vec{r}_1 - \vec{r}_3) \times l_1 l'_1 + \\ & (\vec{r}_2 - \vec{r}_3) \cdot (\vec{r}_2 - \vec{r}_3) \times l_2 l'_2 + \\ & (\vec{r}_1 - \vec{r}_3) \cdot (\vec{r}_2 - \vec{r}_3) \times l_1 l'_2 + \\ & (\vec{r}_1 - \vec{r}_3) \cdot (\vec{r}_2 - \vec{r}_3) \times l'_1 l_2 \end{aligned} \quad (25)$$

Substitution of equation (25) into the first term on the right-hand side of equation (23) results in seven integrals. Each of those is reduced to one of the four independent base integrals given below [5]. The remaining values are obtained using cyclic transformation. First term on the right-hand side of equation (22) only needs the first base integral [5].

With taking into account that

$$a = (\vec{r}_3 - \vec{r}_1) \cdot (\vec{r}_3 - \vec{r}_1), b = (\vec{r}_3 - \vec{r}_1) \cdot (\vec{r}_3 - \vec{r}_2), c = (\vec{r}_3 - \vec{r}_2) \cdot (\vec{r}_3 - \vec{r}_2), \det = \sqrt{a - 2b + c} \quad (26)$$

one has [5]

$$\left(\frac{1}{4A_p^2} \right) \int_p \int_{t_p} \frac{1}{|\vec{r} - \vec{r}'|} d\vec{r}' d\vec{r} = \left[\begin{aligned} & \frac{\log \left(\frac{(a - b + \sqrt{a} \det)(b + \sqrt{a} \sqrt{c})}{(-b + \sqrt{a} \sqrt{c})(-a + b + \sqrt{a} \det)} \right)}{6\sqrt{a}} + \\ & \frac{\log \left(\frac{(b + \sqrt{a} \sqrt{c})(-b + c + \sqrt{c} \det)}{(b - c + \sqrt{c} \det)(-b + \sqrt{a} \sqrt{c})} \right)}{6\sqrt{c}} + \\ & \frac{\log \left(\frac{(a - b + \sqrt{a} \det)(-b + c + \sqrt{c} \det)}{(b - c + \sqrt{c} \det)(-a + b + \sqrt{a} \det)} \right)}{6\det} \end{aligned} \right] \quad (27)$$

$$\left(\frac{1}{4A_p^2} \right) \int_p \int_{t_p} \frac{1}{|\vec{r} - \vec{r}'|} l_1' l_1 d\vec{r}' d\vec{r} = \left[\begin{aligned} & \frac{\log \left(\frac{b + \sqrt{a} \sqrt{c}}{b - c + \sqrt{c} \det} \right)}{40\sqrt{c}} + \frac{\log \left(\frac{-b + c + \sqrt{c} \det}{-b + \sqrt{a} \sqrt{c}} \right)}{40\sqrt{c}} + \\ & \frac{(\sqrt{a} - \sqrt{c})}{120\det^2} + \frac{-(\sqrt{a} \sqrt{c}) + \sqrt{a} \det}{120a^{\frac{3}{2}}} + \\ & \frac{(2a - 5b + 3c) \log \left(\frac{(a - b + \sqrt{a} \det)(c - b + \sqrt{c} \det)}{(b - a + \sqrt{a} \det)(b - c + \sqrt{c} \det)} \right)}{120\det^3} + \\ & \frac{(2a + b) \log \left(\frac{(b + \sqrt{a} \sqrt{c})(a - b + \sqrt{a} \det)}{(-b + \sqrt{a} \sqrt{c})(-a + b + \sqrt{a} \det)} \right)}{120a^{\frac{3}{2}}} \end{aligned} \right] \quad (28)$$

$$\left(\frac{1}{4A_p^2} \right) \int_{t_p} \int_{t_p} \frac{1}{|\vec{r} - \vec{r}'|} l_2' l_1 d\vec{r}' d\vec{r} = \left[\begin{aligned} & \frac{\log\left(\frac{b + \sqrt{a}\sqrt{c}}{b - c + \sqrt{c} \det}\right)}{120\sqrt{c}} + \frac{\log\left(\frac{a - b + \sqrt{a} \det}{-b + \sqrt{a}\sqrt{c}}\right)}{120\sqrt{a}} + \\ & \frac{(2a - 3b + c)\log\left(\frac{a - b + \sqrt{a} \det}{b - c + \sqrt{c} \det}\right)}{120 \det^3} + \\ & \frac{(a - 3b + 2c)\log\left(\frac{-b + c + \sqrt{c} \det}{-a + b + \sqrt{a} \det}\right)}{120 \det^3} + \\ & \frac{-3\sqrt{a} + 3 \det}{120c} + \frac{-3\sqrt{c} + 3 \det}{120a} \\ & + \frac{(3b + 2c)\log\left(\frac{-b + c + \sqrt{c} \det}{-b + \sqrt{a}\sqrt{c}}\right)}{120c^{\frac{3}{2}}} + \\ & + \frac{(2a + 3b)\log\left(\frac{b + \sqrt{a}\sqrt{c}}{-a + b + \sqrt{a} \det}\right)}{120a^{\frac{3}{2}}} \end{aligned} \right] \quad (29)$$

$$\left(\frac{1}{4A_p^2} \right) \int_{t_p} \int_{t_p} \frac{1}{|\vec{r} - \vec{r}'|} l_1' d\vec{r}' d\vec{r} = \left[\begin{aligned} & \frac{-\log\left(\frac{-b + \sqrt{a}\sqrt{c}}{a - b + \sqrt{a} \det}\right)}{24\sqrt{a}} + \frac{\log\left(\frac{b + \sqrt{a}\sqrt{c}}{b - c + \sqrt{c} \det}\right)}{24\sqrt{c}} + \\ & \frac{-(\sqrt{a}\sqrt{c}) + (\sqrt{a} \det)}{24a^{\frac{3}{2}}} + \frac{(a + b)\log\left(\frac{b + \sqrt{a}\sqrt{c}}{-a + b + \sqrt{a} \det}\right)}{24a^{\frac{3}{2}}} + \\ & \frac{\log\left(\frac{a - b + \sqrt{a} \det}{b - c + \sqrt{c} \det}\right)}{24 \det} - \frac{\log\left(\frac{-b + \sqrt{a}\sqrt{c}}{-b + c + \sqrt{c} \det}\right)}{12\sqrt{c}} + \\ & \frac{(\sqrt{a} \det) - (\sqrt{c} \det)}{24 \det^3} + \frac{(a - 3b + 2c)\log\left(\frac{-b + c + \sqrt{c} \det}{-a + b + \sqrt{a} \det}\right)}{24 \det^3} \end{aligned} \right] \quad (30)$$

The remaining integrals are obtained using cyclic transformation. Note that equation (29) is not exactly equation (27) of Ref. [5]. It can be seen that two extra terms in equation (27) of Ref. [5] cancel each other. Four equations (27)-(30) were tested using numerical evaluation

of singular integrals and the singularity extraction procedure, and showed a good agreement with the numerical results.

3.5. Calculation of the near/far field

The scattered fields $\vec{E}^s$ and $\vec{H}^s$ can be expressed in the form [1, 6]

$$\begin{aligned}\vec{E}^s &= -j\omega\vec{A}_s(\vec{r}) - \nabla\Phi_s(\vec{r}) \\ &= \frac{1}{j\omega\epsilon} \left\{ k^2 \int_S \vec{J}_s(\vec{r}') g(\vec{r}, \vec{r}') d\vec{r}' + \nabla \int_S (\nabla \cdot \vec{J}_s(\vec{r}')) g(\vec{r}, \vec{r}') d\vec{r}' \right\} \quad \vec{r}' \text{ on } S\end{aligned}\quad (31)$$

$$\begin{aligned}\vec{H}^s &= \nabla \times \vec{A}_s(\vec{r}) = \nabla \times \int_S \vec{J}_s(\vec{r}') g(\vec{r}, \vec{r}') d\vec{r}' \\ &= \int_S \nabla \times g(\vec{r}, \vec{r}') \cdot \vec{J}_s(\vec{r}') d\vec{r}' \quad \vec{r}' \text{ on } S\end{aligned}\quad (32)$$

at any observation point r in space. Using the expansion for the surface current density (6) one has

$$\vec{E}^s = \sum_{n=1}^N \left\{ \frac{k^2}{j\omega\epsilon} \int_{S_n} \vec{f}_n^s(\vec{r}') g(\vec{r}, \vec{r}') d\vec{r}' + \frac{1}{j\omega\epsilon} \int_S (\nabla \cdot \vec{f}_n^s(\vec{r}')) \nabla_{\vec{r}} g(\vec{r}, \vec{r}') d\vec{r}' \right\} I_n \quad \vec{r}' \text{ on } S \quad (33)$$

$$\vec{H}^s = \sum_{n=1}^N \left\{ \int_S \nabla \times g(\vec{r}, \vec{r}') \cdot \vec{f}_n^s(\vec{r}') d\vec{r}' \right\} I_n \quad \vec{r}' \text{ on } S \quad (34)$$

Similar to the approach to the impedance matrix discussed in section A3, all integrals in curly brackets in equations (33) and (34) are based upon three base integrals over triangular patches

$$\vec{K}_p^i(\vec{r}) = \int_{t_p} \vec{r}_i g(|\vec{r} - \vec{r}'|) d\vec{r}' \quad p = 1, \dots, P \quad (35a)$$

$$\vec{L}_p^i(\vec{r}) = \int_{t_p} \nabla_{\vec{r}} g(|\vec{r} - \vec{r}'|) d\vec{r}' \quad p = 1, \dots, P \quad (35b)$$

$$\vec{M}_p^i(\vec{r}) = \int_{t_p} \nabla_{\vec{r}} \times \vec{r}_i g(|\vec{r} - \vec{r}'|) d\vec{r}' \quad p = 1, \dots, P \quad (35c)$$

to within a constant factor. Here, $\vec{r}_i = \vec{r} - \vec{r}_i$ for any vertex i of patch p . The integrals (35) are found numerically, using either Gaussian integration when

$$D_p(\vec{r}) = \frac{|\vec{r}_c^p - \vec{r}|}{\max_{i=1,2,3}(|\vec{r}_c^p - \vec{r}_i|)} < 15 \quad p = 1, \dots, P \quad (36)$$

or the central-point approximation when the triangular patch is far enough from the observation point.

This approach leads to reasonably accurate results for the near field when the distance from the closest metal patch to the observation point is greater than the size of that patch.

Otherwise, the analytical treatment of singularities in integrals (35) becomes necessary, which is beyond the scope of the present study.

References

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