

Section III. Antenna radiation in time domain

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- Problems**

In the previous section, we have used the fact that the radiated electric field of a dipole is given by Eq. (6). This was actually the major step in establishing the analytical transfer function for the antenna-to-antenna link. How is this equation obtained? How does the dipole radiate in other directions? How do other antennas radiate?

In order to answer these and similar questions, and in order to establish the antenna radiation pattern one needs to study Maxwell's equations for electromagnetic fields. This section only provides an introduction to Maxwell's equations and describes one simple numerical method of their solution. A comprehensive theory of engineering electromagnetics may be found elsewhere¹.

1. Maxwell's equations

Consider an arbitrary (inhomogeneous) medium with electric permittivity ϵ having the units of F/m and with magnetic permeability μ having the units of H/m. We assume that there is no (fictitious) magnetic current density.

The *driving sources* for the electromagnetic fields are given by (generally volumetric) *electric current density* $\vec{J}$ of free charges with the units of A/m², and by volumetric *free charge density* ρ with the units of C/m³. The free charges are free electrons in a metal or free electrons and/or holes in a semiconductor.

Instead of volumetric currents one may consider surface currents (a blade metal dipole) or line current (an infinitesimally thin cylindrical dipole).

¹ C. A. Balanis, *Advanced Engineering Electromagnetics*, Wiley, New York, 1989.
R. F. Harrington, *Time-Harmonic Electromagnetic Fields*, McGraw Hill, New York, 1961
(Reprinted IEEE-Wiley, New York, 2001).

Maxwell's equations for the electric field (or the electric field intensity) $\vec{E}$ [V/m] and for the magnetic field (or the magnetic field intensity) $\vec{H}$ [A/m] in time domain have the form

$$\text{Ampere's law modified by displacement currents} \quad \epsilon \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{H} - \vec{J} \quad (1a)$$

$$\text{Faraday's law} \quad \mu \frac{\partial \vec{H}}{\partial t} = -\nabla \times \vec{E} \quad (1b)$$

$$\text{Gauss' law for electric field} \quad \nabla \cdot \epsilon \vec{E} = \rho \quad (1c)$$

$$\text{Gauss' law for magnetic field (no magnetic charges)} \quad \nabla \cdot \mu \vec{H} = 0 \quad (1d)$$

$$\text{Continuity equation for the electric current (from (1a), (1c))} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \quad (1e)$$

Note: If $\vec{J}$ and ρ are equal to zero, a solution to Eqs. (1) in unbounded space is given by $\vec{E} = \vec{H} = 0$. In other words, there are no fields in free space without the *sources*: the electric charges and the electric currents. On the other hand, if $\vec{E} = \vec{H} = 0$ everywhere in space then $\vec{J}$ and ρ are equal to zero too.

Eq. (1d) has a general solution in the form

$$\nabla \cdot \mu \vec{H} = 0 \Rightarrow \mu \vec{H} = \nabla \times \vec{A} \quad (1f)$$

where $\vec{A}$ is a new function - the *magnetic vector potential* with the units of V·s/m. Eq. (1f) follows from the vector calculus identity $\nabla \cdot (\nabla \times \vec{A}) = 0$, which is valid for any function $\vec{A}(\vec{r}, t)$.

Question: What else except for Maxwell's equations do we need to know in order to establish antenna radiation characteristics and be able to model antennas and other RF circuits?

Answer: Boundary conditions to Maxwell's equations are critical for both analytical and numerical methods. They are briefly reviewed in the next subsection.

2. Boundary conditions

2.1. General material interface

On the interface between two magnetic materials or two dielectric materials, or both, the following boundary conditions are employed (medium I and medium II are those from Fig. 1) in time domain:

$$\vec{n} \cdot (\mu_2 \vec{H}_2 - \mu_1 \vec{H}_1) = 0 \quad \text{normal component of the } B\text{-field is continuous} \quad (2a)$$

$$\vec{n} \cdot (\epsilon_2 \vec{E}_2 - \epsilon_1 \vec{E}_1) = 0 \quad \text{normal component of the } D\text{-field is continuous} \quad (2b)$$

$$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = 0 \quad \text{tangential component of the } H\text{-field is continuous} \quad (2c)$$

$$\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \quad \text{tangential component of the } E\text{-field is continuous} \quad (2d)$$

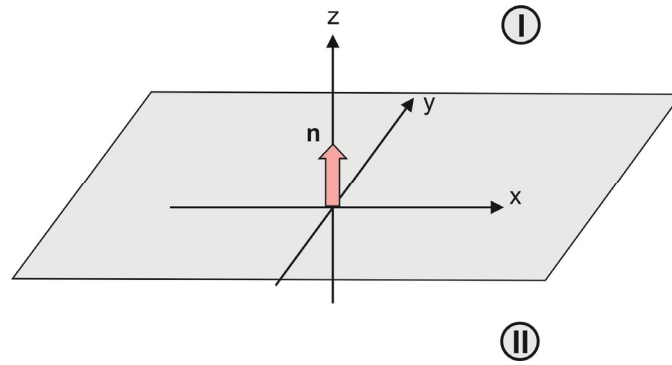


Fig. 1. Problem geometry to boundary conditions at a media interface.

2.2. Metal-dielectric (metal-air) interface

On a metal surface S of a *perfect* electric conductor or PEC (represented by medium II in Fig. 1), the tangential component of the total electric field vanishes,

$$\vec{E}_{\text{tan}} = \vec{n} \times \vec{E} = 0 \quad (2e)$$

where *tan* denotes the tangential component of the E -field that is parallel to the surface. The proof is based on Eq. (2d) which uses the fact that $\vec{E}_2 = 0$ in a perfect conductor. Otherwise, an infinite current would flow there, due to infinite conductivity.

Note: The PEC boundary condition (2e) is a good approximation to reality for antennas operating at 10 GHz and lower. The PEC boundary condition implies *no loss*. At higher frequencies, the concept of surface impedance should be used.

The normal component of the electric field could have any nonzero values associated with the surface charge. There is no electric or magnetic field within the perfect electric conductor. Then, the boundary condition for the magnetic field is trivially obtained from the Ampere's law Eq. (1a) using Eq. (2a)

$$(\nabla \times \vec{H} - \vec{J})_{\text{tan}} = 0 \quad (2f)$$

This boundary condition is used less often than the boundary condition given by Eq. (2e).

2.3 Metal surface of finite conductivity, corrugated metal surface, rough metal surface, etc.

To establish boundary conditions for a non-perfect metal surface, the phasor form of Maxwell's equations and the phasors of field quantities are often used. The original time domain form is less appropriate, even though it is also applicable. The phasors are considered further in the next section. However, we present here some results here for completeness.

In the phasor form or, which is the same, in the frequency domain, one introduces the time dependence in the form of a factor $\exp(j\omega t)$ for all the field quantities. The factor $\exp(j\omega t)$ is then cancelled in every equation. The phasors will be denoted by the same (but bold) letters as the original fields.

Consider a plane wave incidence on the metal surface of a finite conductivity as shown in Fig. 2. The plane wave incidence plane is the xz -plane. The whole surface can be characterized by a *surface impedance* $\mathbf{Z}_s$.

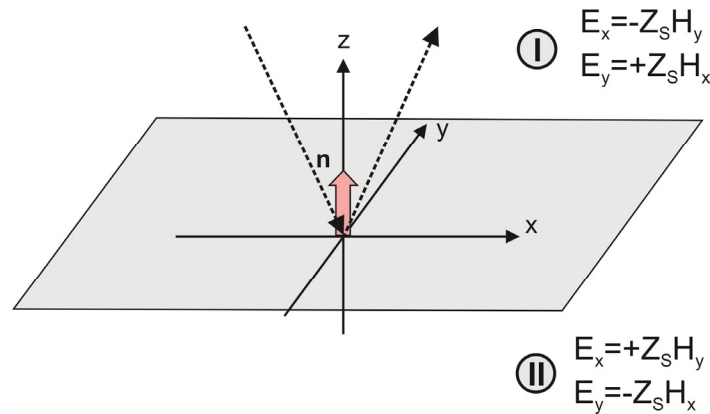


Fig. 2. The (isotropic) surface impedance concept.

This literally means that the tangential components of both fields are related by²

² P. S. Kildal, "Artificially soft and hard surfaces in electromagnetics," *IEEE Trans. Antennas Propagation*, vol. 38, no. 10, pp. 1537-1544, Oct. 1990.

$$\vec{E}_{\text{tan}} = \mathbf{Z}_S \vec{n} \times \vec{H} = \mathbf{Z}_S \vec{H}_{\text{tan}} \quad (2g)$$

where $\vec{n}$ is the *outer* normal to the surface (directed to half space I in Fig. 2).

Question: Which units does the surface impedance have?

Answer: Eq. (2g) predicts the units of ohms. However, for a rectangular waveguide a surface impedance termination coincides with a lumped-circuit termination when the waveguide has a *square* area – the reason why the surface impedance is also measured in *ohms per square* – even though the real units for the surface impedance are indeed ohms.

Question: Which value does the surface impedance have for a perfect metal conductor?

Answer: Comparing Eqs. (2e) and (2g) we obtain $\mathbf{Z}_S = 0$.

Eq. (2g) is further expanded in the form

$$\mathbf{E}_x \hat{x} + \mathbf{E}_y \hat{y} = -\mathbf{Z}_S \mathbf{H}_y \hat{x} + \mathbf{Z}_S \mathbf{H}_x \hat{y} \quad (2h)$$

or, which is the same,

$$\mathbf{E}_x = -\mathbf{Z}_S \mathbf{H}_y, \quad \mathbf{E}_y = \mathbf{Z}_S \mathbf{H}_x \quad (2i)$$

On the other hand, when looking from half space II (the incident wave is coming from the bottom to the top), the surface normal direction is changed to the opposite one, so that one has the opposite signs in both Eqs. (2i). Thus, whilst the surface impedance indeed remains the same in either case, the field relation changes its sign depending on the incidence direction – see Fig. 2³.

The simplest case of the isotropic impedance surface is an imperfect metal conductor with a finite skin layer: the surface impedance becomes

³ A similar situation takes place for the wave impedance in a plane wave: when $\vec{E}, \vec{H}, \vec{k}$ are directed along x, y, z , it is $E_x = \eta H_y$, but $E_x = -\eta H_y$ otherwise, when k is directed along $-z$.

$$\mathbf{Z}_s = \sqrt{\frac{j\omega\mu_0}{\sigma}} \quad (2j)$$

where σ is the metal conductivity. This surface impedance takes into account the skin layer losses. We remember that the depth δ of the skin layer in a metal conductor is given by $\delta = \sqrt{2/\omega\mu_0\sigma}$. Interestingly, the surface impedance of a metal corrugation in the direction perpendicular to the wave propagation direction is given by

$$\mathbf{Z}_s \equiv j\eta \frac{g}{g+t} \tan kd \quad (2k)$$

where t , is the tooth width, g is the tooth spacing, and d is the corrugation depth⁴.

3. Analytical solution to Maxwell's equations in time domain

The most important solution to Maxwell's is a *plane wave* shown in Fig. 3. In Fig. 3, the plane wave propagates with the speed of light in the y direction and has the vertical component of the electric field, $\mathbf{E}_z$, and the horizontal component of the magnetic field, $\mathbf{H}_x$. It very much resembles an ocean wave coming toward you when you are on a beach. The only one (and principal) difference is that the plane wave has an extra field component (the H-field).

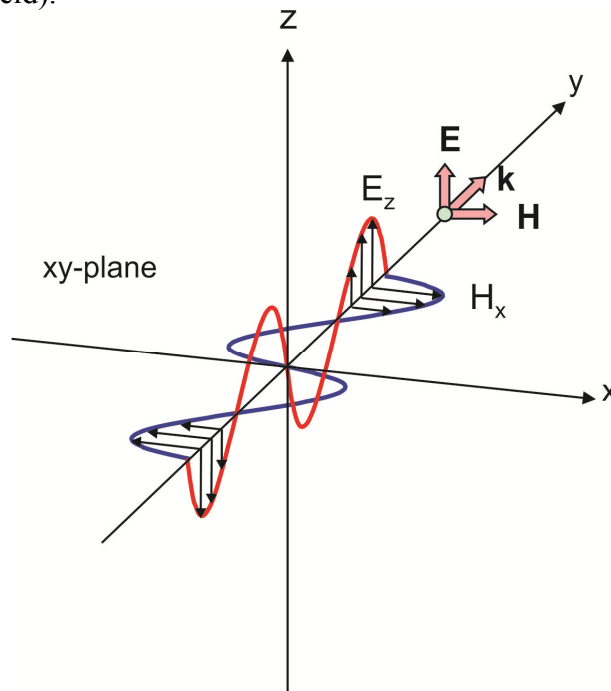


Fig. 3. Plane electromagnetic wave that propagates along the x-axis.

⁴ C. A. Balanis, *Antenna Theory. Analysis and Design*, Wiley, New York, 2005 3rd ed., p. 789.

Note: The plane wave is so called because it has a plane phase front; it is not spherical or cylindrical wave. In Fig. 3, the phase front is given by a plane that is *parallel* to the xy -plane.

Any radiated field at large distances from the source is locally represented by plane waves as explained in the next section. Such plane waves could be generated using different means, for example, by a GPS satellite, or by a base station antenna, or by your cellphone far away from it, etc.

The plane wave is an one-dimensional solution to Maxwell's equations (1). With reference to Fig. 3, Maxwell's equations in one dimension become

$$\text{Ampere's law} \quad \varepsilon \frac{\partial E_z}{\partial t} + \frac{\partial H_x}{\partial y} + J_z = 0 \quad (3a)$$

$$\text{Faraday's law} \quad \mu \frac{\partial H_x}{\partial t} + \frac{\partial E_z}{\partial y} = 0 \quad (3b)$$

So far, no other equations become necessary.

For simplicity, we assume that μ and ε are constant. Then, one can prove by direct substitution that one possible solution to Eqs. (3), in a domain where the excitation current J_z is zero, has the form

$$E_z(y, t) = E_{z0} \cos \frac{\omega}{c} (y - ct) \quad (4a)$$

$$H_x(y, t) = \frac{1}{\eta} E_z(y, t) \quad (4b)$$

where

$$c = \frac{1}{\sqrt{\varepsilon \mu}} \quad (4c)$$

is *speeds of light* in a medium with dielectric permittivity ε and magnetic permeability μ (3×10^8 m/s in vacuum or air) and

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \quad (4d)$$

is the *characteristic impedance* of the same medium with the units of ohms (377Ω in vacuum or air). This solution in the form of a sinusoidal plane wave is shown in Fig. 3.

Note: It will be shown in what follows that Eqs. (3), (4) are *identical* to the equations for a RF transmission line if we replace the electric field by line voltage, the magnetic field - by line current, the dielectric constant - by line capacitance per unit length, and the magnetic permeability – by line inductance per unit length, i.e.

$$\begin{aligned} E &\rightarrow V \\ H &\rightarrow I \\ \varepsilon &\rightarrow C \\ \mu &\rightarrow L \end{aligned} \tag{5}$$

This fact underscores a deep interconnection between the circuit quantities and the field quantities and allows us to better “visualize” the E- and H-fields as a “distributed” voltage and “distributed” current.

4. Numerical solution to Maxwell’s equations in time domain

Unfortunately, the analytical solution to Eqs. (3) may be seldom obtained in a closed simple form, even in one dimension. A case in point is plane wave propagation in an inhomogeneous dielectric material.

Today, the standard method for the solution of Maxwell’s equations is a numerical method. Although many important analytical solutions to Maxwell’s equations have been obtained during the last 100 or so years, their time now passed.

When solving Eqs. (3) numerically we have at least three choices:

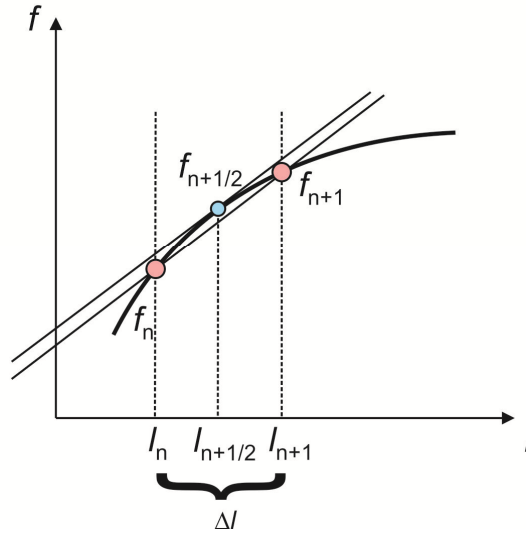
1. Reduce them to integral equations and solve using the *Method of Moments* (MoM) in one dimension, either in time domain or in frequency domain.
2. Introduce linear (or higher-order) finite elements over the length of the trace (in the y-direction) and use the Finite Element Method (FEM), either in time domain or in frequency domain.
3. Directly approximate the time- and space-derivatives in Eqs. (3) by finite differences and use the *Finite Differences Time Domain* (FDTD) method.

All the full-wave numerical EM software essentially uses one of those methods or a combination of them. In this section, we prefer to use the last FDTD method since it has a great simplicity and could be easily implemented.

4.1 Finite-difference time-domain method

The “heart” of the FDTD method is shown in Fig. 4. This is a finite-difference approximation of any spatial derivative (or slope) in Eqs. (3).

Second-order accurate slope approximation in space (time)

Fig. 4. Slope approximation at the midpoint between nodes n and $n+1$.

We intend to approximate the slope of a function $f(l)$ of argument l at a point $f(l_{n+1/2})$, i.e. exactly in the middle between two “regular” nodes $f(l_n)$ and $f(l_{n+1})$ separated by Δl . As follows from Fig. 4, a result that is second-order accurate (the error is less than $O(\Delta l^2)$) has the form:

$$\frac{df}{dl}(l_{n+1/2}) = \frac{f_{n+1} - f_n}{\Delta l} \quad (6a)$$

Eq. (6a) approximates both time and spatial derivatives in Eqs. (3).

4.2 Grid in space/time

In order to use Eq. (6a) we must introduce a *grid* (or a *mesh*) within a spatial interval $y \in [0, D]$ under interest. This grid will include both regular and intermediate nodes – see Fig. 5 that follows, i.e.

$$Y_E = [0, \Delta l, 2\Delta l, 3\Delta l, \dots, D] \quad \text{regular nodes} \quad (6b)$$

$$Y_H = [\Delta l/2, 3\Delta l/2, 5\Delta l/2, \dots, D - \Delta l/2] \quad \text{centers of intervals} \quad (6c)$$

Here, Δl is the *grid size*. We also introduce the grid in a time interval $t \in [0, T]$ under interest. This grid will include both regular and intermediate nodes too – see Fig. 5, i.e.

$$T_E = [0, \Delta t, 2\Delta t, 3\Delta t, \dots, D] \quad \text{regular nodes} \quad (6d)$$

$$T_H = [\Delta t/2, 3\Delta t/2, 5\Delta t/2, \dots, D - \Delta t/2] \quad \text{centers of intervals} \quad (6e)$$

The grids in Eqs. (6) are called *uniform* since they keep the same discretization interval (grid size Δl or Δt) for the entire domain.

4.3 Staggered grid

It makes sense to rewrite Eqs. (3) without the excitation current in the form

$$\begin{aligned}\varepsilon \frac{\partial E}{\partial t} + \frac{\partial H}{\partial y} &= 0 \\ \mu \frac{\partial H}{\partial t} + \frac{\partial E}{\partial y} &= 0\end{aligned}\tag{7}$$

i.e. omit indexes z and x for the fields, respectively. Those equations are so structured that they are ideal for the so called *staggered grid* where E and H are given at different points both in space and in time.

Namely, the E -field in space is given at the regular (or integer) grid points Y_E whereas the H -field in space is given at the intermediate or interleaving nodes Y_H - see Eqs. (6b, c). Similarly, the E -field in time is given at the regular (or integer) grid points T_E whereas the H -field in time is given at the intermediate or interleaving nodes T_H - see Eqs. (6d, e).

Note: The staggered grid is a de facto standard in numerical time-domain electromagnetics, but it is less common in other disciplines such as fluid mechanics, elasticity theory, solid state mechanics, etc.

If we assign index k to spatial nodes and index n – to temporal nodes, then one has

$$E \rightarrow E_k^n = E(y_k, t_n), \quad y_k = (k-1)\Delta l, \quad t_n = (n-1)\Delta t\tag{8a}$$

$$H \rightarrow H_{k+1/2}^{n+1/2} = E(y_{k+1/2}, t_{n+1/2}), \quad y_{k+1/2} = (k-1/2)\Delta l, \quad t_{n+1/2} = (n-1/2)\Delta t\tag{8a}$$

with reference to Fig. 5 that follows.

Question: Why is it $y_k = (k-1)\Delta l$, $t_n = (n-1)\Delta t$ in Eq. (8a), but not $y_k = k\Delta l$, $t_n = n\Delta t$?

Answer: In MATLAB and some other software languages array indexes start with 1, not with zero, so that we must subtract one to include the value $y = 0$ or $t = 0$.

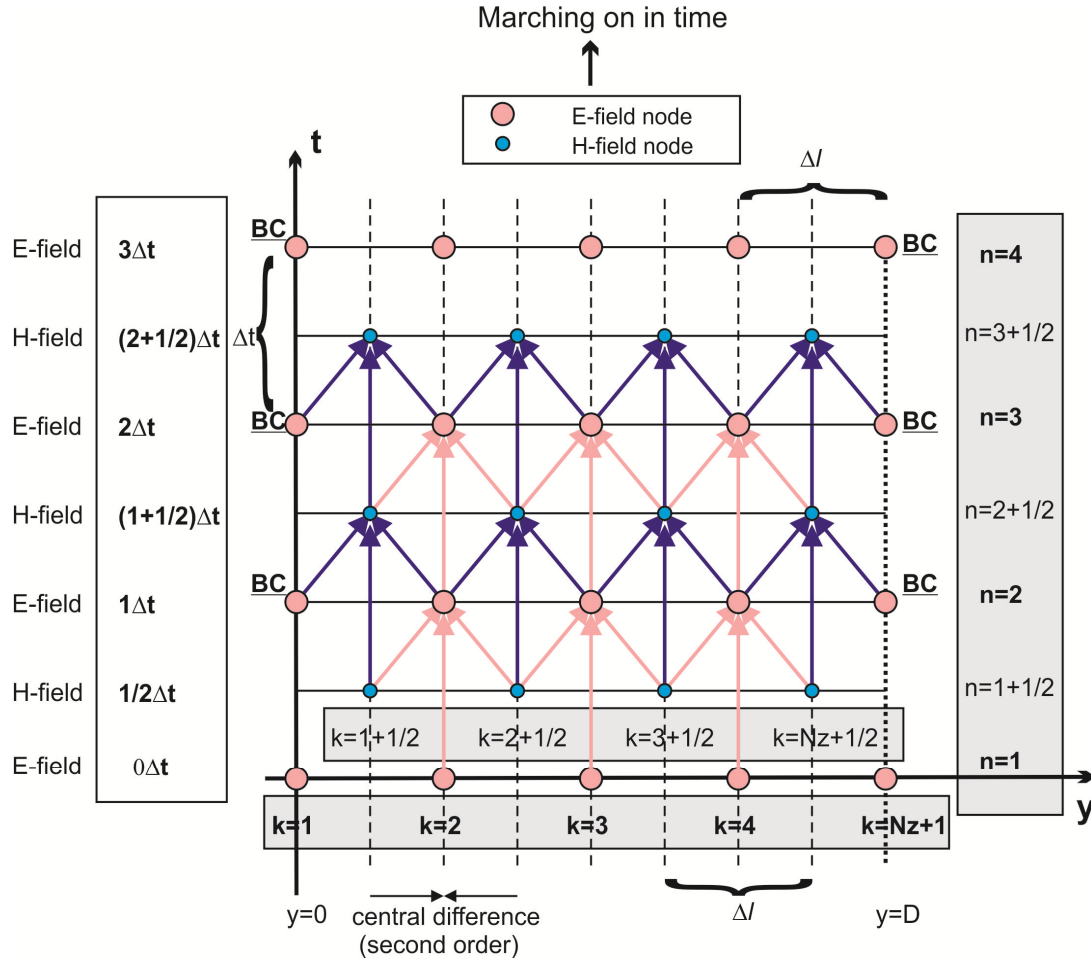


Fig. 5. The concept of a staggered grid and marching on in time.

4.3 Marching on in time

Using Eq. (6a) for derivative approximations and the notations given by Eqs. (8) one obtains from Eqs. (7) the corresponding *finite-difference* equations⁵

$$\begin{aligned} \epsilon \frac{\partial E}{\partial t} + \frac{\partial H}{\partial y} = 0 &\Rightarrow \epsilon \frac{E_k^{n+1} - E_k^n}{\Delta t} + \frac{H_{k+1/2}^{n+1/2} - H_{k-1/2}^{n+1/2}}{\Delta y} = 0 \Rightarrow E_k^{n+1} = E_k^n - \frac{\Delta t}{\epsilon \Delta y} (H_{k+1/2}^{n+1/2} - H_{k-1/2}^{n+1/2}) \\ \mu \frac{\partial H}{\partial t} + \frac{\partial E}{\partial y} = 0 &\Rightarrow \mu \frac{H_{k+1/2}^{n+3/2} - H_{k+1/2}^{n+1/2}}{\Delta t} + \frac{E_{k+1}^{n+1} - E_k^{n+1}}{\Delta y} = 0 \Rightarrow H_{k+1/2}^{n+3/2} = H_{k+1/2}^{n+1/2} - \frac{\Delta t}{\mu \Delta y} (E_{k+1}^{n+1} - E_k^{n+1}) \end{aligned} \quad (9)$$

⁵ K. S. Yee, "Numerical solution of initial boundary value problem involving Maxwell's equations in isotropic media," *IEEE Trans. Antennas and Propagation*, vol. 14, 1966, pp. 302-307.

that are second-order accurate in space and in time. The method of solution is explained in Fig. 5; it is often called “marching on in time” and may be described as follows:

1. We find all E-field values in space at time step $n = 2$, $t = \Delta t$ using E-field values at the previous time step $n = 1$, $t = 0$ and H-field values at the intermediate time step $n = 1/2$, $t = 1/2\Delta t$.
2. We find all H-field values in space at the intermediate time step $n = 1 + 1/2$, $t = \Delta t + \Delta t/2$ using H-field values at the previous intermediate time step $n = 1/2$, $t = \Delta t/2$ and E-field values already found at the previous time step $n = 2$, $t = \Delta t$.
3. We find all E-field values in space at time step $n = 3$, $t = 2\Delta t$ using E-field values at the previous time step $n = 2$, $t = \Delta t$ and H-field values already found at the intermediate time step $n = 1 + 1/2$, $t = \Delta t + \Delta t/2$.
4. We find all H-field values ..., etc.

The method thus remain "bootstrapping" - see Fig. 5. It is sometimes called exactly in the same way.

Note: Two bottom rows in Fig. 5 are found from using *initial conditions*; the leftmost and rightmost columns in Fig. 5 correspond to *boundary conditions*. The boundary conditions are most important. In examples that follow we will use the so-called *impedance boundary conditions*. They are given by either Eq. (2g) written in time domain or by Eq. (4b), where $\eta = Z_s$ is now the surface impedance that is independent of frequency. One thus has

$$H = \pm \frac{1}{\eta} E \quad (9^*)$$

where sign + corresponds to the right boundary and sign – corresponds to the left boundary. When the surface impedance, η , is that of free space, the boundary is the *absorbing boundary* – it passes the plane wave through. When the surface impedance is zero, the boundary is a perfect metal – it reflects the plane wave. When the surface impedance has another nonzero value, the boundary is a certain dielectric or magnetic material. The impedance boundary conditions are equivalent to termination of a transmission line with certain resistances.

Example 1: Compile a MATLAB script that solves Eqs. (9) augmented with free-space impedance boundary conditions for the trace with the total length of 600mm. The field source is given by $E(t, y = 0) = 2 \sin \omega t$ [V/m] at 1 GHz.

Solution: A MATLAB script that completely solves the problem follows:

```
% EM data
epsilon = 8.85418782e-012; % vacuum, F/m
mu       = 1.25663706e-006; % vacuum, H/m
c        = 1/sqrt(epsilon*mu); % vacuum, m/s
eta       = sqrt(mu/epsilon); % vacuum, Ohm
%-----
% Trace data
D        = 600.0e-3; % Trace length, m
Nl       = 300; dl = D/Nl; % Number of spatial cells and spatial step
lE       = [0:dl:D]; % Y-coordinates for the E-field nodes, Nl+1 total
lH       = [dl/2:dl:D-dl/2]; % Y-coordinates for the H-field nodes, Nl total
%-----
dt        = dl/c; % magic time step
T         = 10e-9; % observation time, s
NT = round(T/dt); t = [0: dt: NT*dt]; % Number of time steps and time vector
delta     = dt/dl;
%-----
% FDTD - marching on in time; initialization:
Hnext     = zeros(1, Nl+0); % fourth layer in the series of four (n+3/2)
Enext     = zeros(1, Nl+1); % third layer in the series of four (n+1)
Hpast     = zeros(1, Nl+0); % second layer in the series of four (n+1/2)
Epast     = zeros(1, Nl+1); % first layer in the series of four (n+0)
indv      = 2:Nl; % batch process in space (voltage or electric field)
indi      = 1:Nl; % batch process in space (current or magnetic field)
%-----
% Electric field source and impedance boundary conditions
f = 1e9; % Frequency of interest
Eleft = 2*sin(2*pi*f*t); % Electric field source, left
Eright = 0*sin(2*pi*f*t); % Electric field source, right
Rleft = eta; % Surface impedance, left boundary
Rright = eta; % Surface impedance, right boundary
Blinv = 1/(Rleft*epsilon/delta + 1); % BC coefficient -left
Bldir = (Rleft*epsilon/delta - 1); % BC coefficient -left
B2inv = 1/(Rright*epsilon/delta + 1); % BC coefficient -right
B2dir = (Rright*epsilon/delta - 1); % BC coefficient -right

e1 = 1; h1 = 1; % Coefficients to finite-difference Eqs.
e2 = dt/(dl*epsilon); h2 = dt/(dl*mu); % Coefficients to finite-difference Eqs.
%-----
% Main loop - "bootstrapping"
n = 1;
while n <= NT-1
    Enext(1) = Blinv*( Bldir*Epast(1) - 2*Rleft*Hpast(1) + Eleft(n) +Eleft(n+1));
    Enext(Nl+1) = B2inv*( B2dir*Epast(Nl+1) + 2*Rright*Hpast(Nl) + Eright(n)+Eright(n+1));
    Enext(indv) = e1*Epast(indv) - e2*(Hpast(indv+0) - Hpast(indv-1));
    Hnext(indi) = h1*Hpast(indi) - h2*(Enext(indi+1) - Enext(indi-0));
    Epast = Enext; Hpast = Hnext; n = n + 1;
    %-----
    string = strcat(num2str(1e9*t(n)), ' ns');
    x = [min(lE) max(lE)]; y = [1 1];
    plot(lE, Enext, 'r', 'LineWidth', 2); grid on;
    line(x, y, 'LineWidth', 2); line(x, -y, 'LineWidth', 2);
    axis([min(lE), max(lE) -3 +3]); title (strcat('Electric field (V/m) at t=',string))
    xlabel('distance, m'); ylabel('E, V/m'); drawnow;
end
```

Note: One mentions that the entire algorithm of Eq. (9) is given in example 1 by two underlined lines of the MATLAB code. The rest of the script includes parameter initialization, constant definitions, source definition, and the boundary conditions.

5. Equivalence of transmission line equations and 1D Maxwell's equations

5.1. TL equations vs. Maxwell's equations

A typical transmission line is shown in Fig. 6. It is characterized by a distributed capacitance per unit length, C , as well as by a distributed inductance, L , and a resistance per unit length, R .

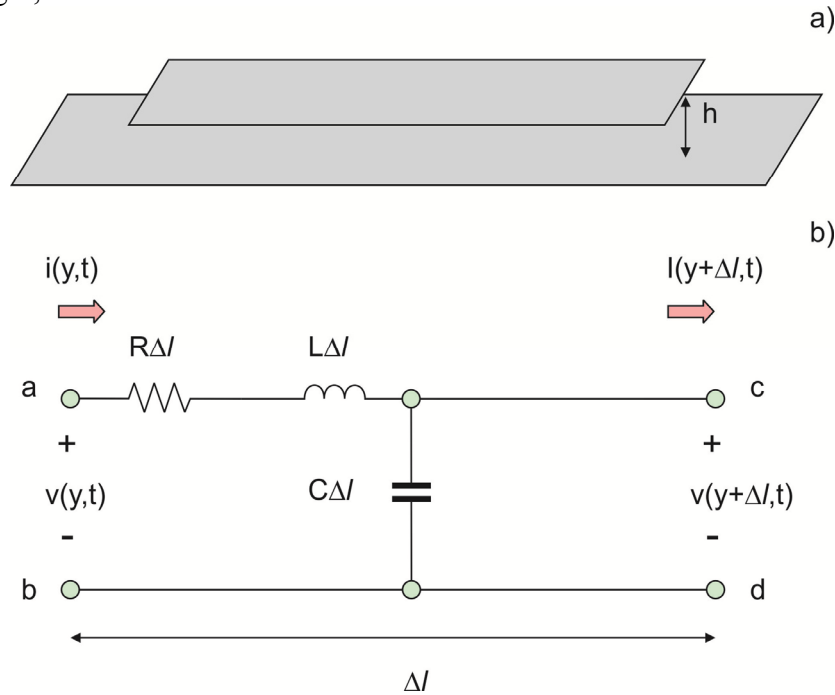


Fig. 6. Transmission line and its equivalent circuit. Line conductance or dielectric loss is ignored. Line resistance or metal loss is included into consideration.

The problem now involves not only the temporal dependence (as in low-frequency AC circuits), but also a spatial dependence since both voltage and current change along the transmission line. As a result, we have a *distributed* electric circuit or an RF circuit.

Transmission line equations for the line voltage $v(y,t)$ (voltage between the top conductor and the ground plane) and the line current $i(y,t)$ (current through the top conductor) are obtained by applying the KVL and the KCL to the circuit shown in Fig. 6b. By KVL for the *abcd* loop,

$$-v(y,t) + R\Delta l i(y,t) + L\Delta l \frac{\partial i}{\partial t} + v(y + \Delta l, t) = 0 \quad (10a)$$

By KCL for the top node,

$$-i(y + \Delta l, t) + i(y, t) = C\Delta l \frac{\partial v}{\partial t} \quad (10b)$$

Dividing these two equations by Δl and grouping the terms, we obtain,

$$\frac{v(y + \Delta l, t) - v(y, t)}{\Delta l} = -Ri(y, t) - L \frac{\partial i}{\partial t} \quad (10c)$$

$$\frac{i(y + \Delta y, t) - i(y, t)}{\Delta l} = -C \frac{\partial v}{\partial t} \quad (10d)$$

or, in terms of spatial derivatives,

$$\frac{\partial v}{\partial y} = -Ri(y, t) - L \frac{\partial i}{\partial t} \quad (10e)$$

$$\frac{\partial i}{\partial y} = -C \frac{\partial v}{\partial t} \quad (10f)$$

Eqs. (10e, f) are the celebrated partial differential equations for the transmission line. The line resistance per unit length, R , is often ignored. Therefore, one finally obtains quite a symmetric result, i.e.

$$C \frac{\partial v}{\partial t} + \frac{\partial i}{\partial y} = 0 \quad (11a)$$

$$L \frac{\partial i}{\partial t} + \frac{\partial v}{\partial y} = 0 \quad (11b)$$

Indeed, this result coincides with 1D Maxwell's equations (7) rewritten here once more for completeness:

$$\epsilon \frac{\partial E}{\partial t} + \frac{\partial H}{\partial y} = 0$$

$$\mu \frac{\partial H}{\partial t} + \frac{\partial E}{\partial y} = 0$$

if one replaces the electric field by line voltage, the magnetic field - by line current, the dielectric constant - by line capacitance per unit length, and the magnetic permeability - by line inductance per unit length.

5.2. Line termination/Impedance boundary conditions for Maxwell's equations

For a single-conductor transmission line system shown in Fig. 7, the corresponding boundary conditions (terminal conditions) read

$$\begin{aligned} v(t, y = 0) &= V_G(t) - R_G i(t, y = 0) \\ v(t, y = D) &= R_L i(t, y = D) \end{aligned} \quad (12)$$

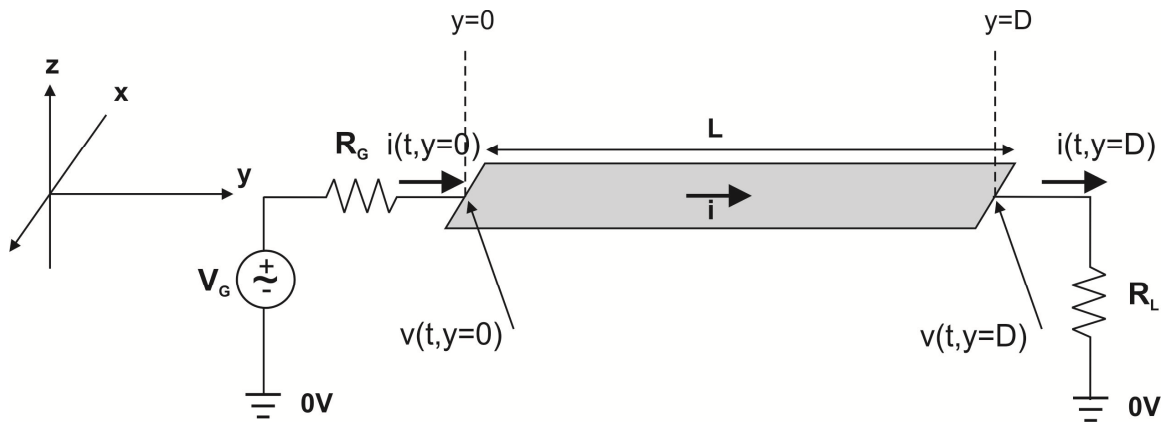


Fig. 7. A generic terminated transmission line. Left - line source; right - line termination.

Variations of boundary conditions given by Eqs. (12) are possible.

To implement those terminal conditions we shall use an approach developed by C. R. Paul in 1994⁶. That approach implies a continuation of the line toward negative y , by $-\Delta l/2$, and then physically imposing the condition of zero current at the end of such a continuation. This approach is widely used today⁷; its idea is shown in Fig. 8 above. Such an approach is exact for a two-conductor line (a microstrip line) and provides a very good approximation for a multi-conductor line.

Thus, a line is extended by $-\Delta l/2$ to the left at the source and by $+\Delta l/2$ to the right at the termination. Since the approach becomes identical in both cases, only the source cell will be considered further. The first step is to replace the generator and the load by its Norton equivalent - see Fig. 8a. Then, the new section of the line with the total length, $+\Delta l$, centered at $y = 0$, will have the form shown in Fig. 8b.

⁶ C. R. Paul, "Incorporation of terminal constraints in the FDTD analysis of transmission lines", *IEEE Trans. Electromagnetic Compatibility*, vol. 36, no. 2, May 1994, pp. 85-91.

⁷ See, for example, A. Taeb, A. Abdipour, and A. Mohhamadi, "FDTD analysis of the lossy coupled transmission lines loaded by nonlinear devices," *Asia-Pacific Microwave Conference Proceedings*, 2005, vol. 5, 4-7 Dec. 2005, 4 pages.

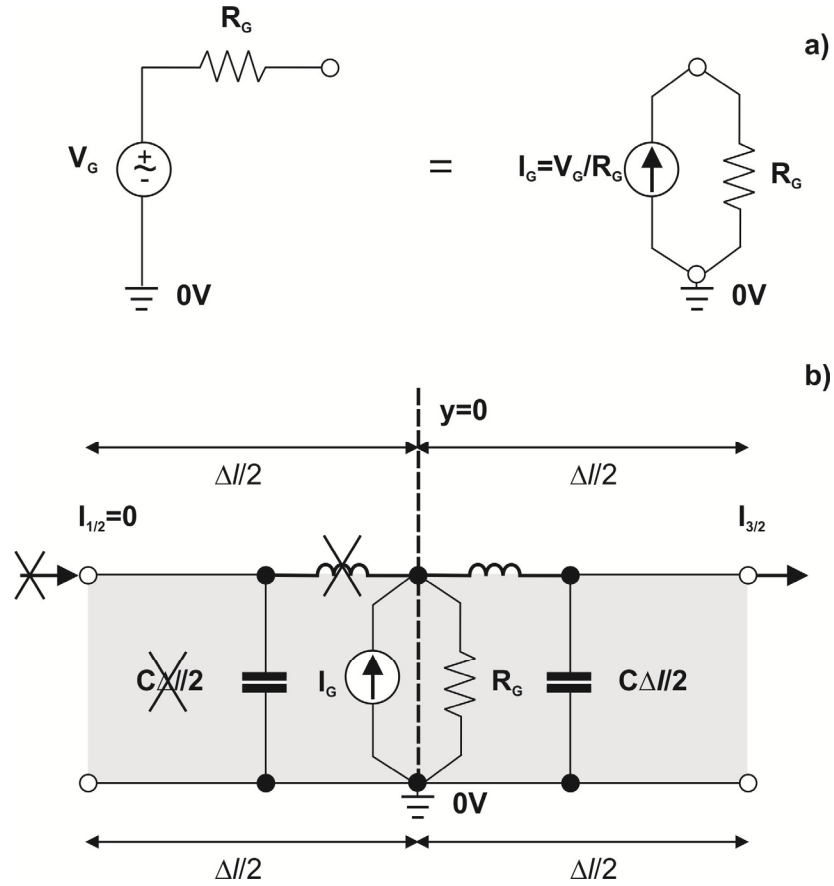


Fig. 8. Source boundary termination section of a TL extended into an imaginary line section of the length $\Delta l/2$.

It follows from Fig. 8 that, for the "extended" unit cell of length L , the distributed capacitance C and distributed inductance L could be replaced by their "average" values:

$$\begin{aligned} C &\rightarrow C/2 \\ L &\rightarrow L/2 \end{aligned} \quad (13)$$

since the distributed reactance values for the imaginary line continuation of length $\Delta l/2$ are zero.

The current source and the shunt resistance imposed at the center of the cell are then considered as distributed loads over the entire unit cell. Eq. (11a) (the KCL on a line) is now rewritten in the form

$$C \frac{\partial v(t, y)}{\partial t} + \frac{\partial i(t, y)}{\partial y} = 0 \rightarrow \frac{C}{2} \frac{\partial v(t, y)}{\partial t} + \frac{\partial i(t, y)}{\partial y} + Gv = I_F \quad (14)$$

where

$$G = \frac{1}{R_G \Delta l}, \quad I_F = \frac{V_G}{R_G \Delta l} \quad (15)$$

are distributed conductance and the distributed density of current sources, respectively.

5.3. Finite differences

Only one voltage node needs to be recalculated, only at $k=1$ in Fig. 5. The second-order discretization of Eq. (14) over the staggered grid gives

$$\frac{C}{2} \frac{V_1^{n+1} - V_1^n}{\Delta t} + \frac{I_{3/2}^{n+1/2} - \underbrace{I_{1/2}^{n+1/2}}_{=0!}}{\Delta l} + G \frac{V_1^{n+1} + V_1^n}{2} = I_F = \frac{V_G^{n+1} + V_G^n}{2 R_G \Delta l} \quad (16)$$

One notes that time averaging has been used to calculate the voltage term and the right-hand side of Eq. (14). Such an averaging preserves the second-order accuracy of the entire numerical routine. From Eq. (16), the boundary voltage at $k=1$ can be expressed. After combining the similar terms, one has

$$V_1^{n+1} = \left[\frac{\Delta l}{\Delta t} R_G C + 1 \right]^{-1} \left\{ \left[\frac{\Delta l}{\Delta t} R_G C - 1 \right] V_1^n - 2 R_G I_{3/2}^{n+1/2} + V_G^{n+1} + V_G^n \right\} \quad (17)$$

The load boundary condition at $y = D$ is identical to Eq. (17), except for the following two replacements to be made in Eq. (17):

$$\begin{aligned} R_L &\rightarrow -R_G \\ V_G &\rightarrow 0 \end{aligned} \quad (18)$$

The minus sign appears due to the opposite voltage drop through the load resistance versus ground as compared to the source resistance.

Note: Different types of the terminal receiving load, e.g. a nonlinear Shockley diode, or a Shockley diode in series with a capacitor (an envelope detector AM circuit connected to an antenna), or etc. could be modeled in a similar fashion.

Note: Accurate implementation of a boundary source or an impedance boundary condition for Maxwell's equations in 1D has been done in example 1 exactly in the same way.

Example 2: Convert the MATLAB script for example 1 to the script for a terminated transmission line with $C = 1.22 \times 10^{-12}$ F/m, $L = 3.02 \times 10^{-7}$ H/m (line impedance is 50Ω).

Solution: We modify the script according to Eqs. (5) and obtain

```
% EM data
C = 1.22e-010; % Line capacitance per unit length, F
L = 3.02e-007; % Line inductance per unit length, H
c = 1/sqrt(C*L); % Line phase speed, m/s
eta = sqrt(L/C); % Line characteristic impedance, Ohm
%-----
% Trace data
D = 600.0e-3; % Trace length, m
Nl = 300; dl = D/Nl; % Number of spatial cells and spatial step
lV = [0:dl:D]; % Y-coordinates for voltage nodes, Nl+1 total
lI = [dl/2:dl:D-dl/2]; % Y-coordinates for current nodes, Nl total
%-----
dt = dl/c; % magic time step
T = 10e-9; % observation time, s
NT = round(T/dt); t = [0: dt: NT*dt]; % Number of time steps and time vector
delta = dt/dl;
%-----
% FDTD - marching on in time; initialization
Inext = zeros(1, Nl+0); % fourth layer in the series of four (n+3/2)
Vnext = zeros(1, Nl+1); % third layer in the series of four (n+1)
Ipast = zeros(1, Nl+0); % second layer in the series of four (n+1/2)
Vpast = zeros(1, Nl+1); % first layer in the series of four (n+0)
indv = 2:Nl; % batch process in space (voltage or electric field)
indi = 1:Nl; % batch process in space (current or magnetic field)
%-----
% Voltage source and impedance boundary conditions
f = 1e9; % Frequency of interest
Vleft = 2*sin(2*pi*f*t); % Generator voltage source, left
Vright = 0*sin(2*pi*f*t); % Load voltage source (if any), right
Rleft = 50; % Generator resistance, left boundary
Rright = 50; % Load resistance, right boundary

Blinv = 1/(Rleft*C/delta + 1); % BC coefficient -left
Bldir = (Rleft*C/delta - 1); % BC coefficient -left
B2inv = 1/(Rright*C/delta + 1); % BC coefficient -right
B2dir = (Rright*C/delta - 1); % BC coefficient -right

e1 = 1; e2 = dt/(dl*C);
h1 = 1; h2 = dt/(dl*L);

% Main loop - "bootstrapping"
n = 1;
while n <= NT-1
    Vnext(1) = Blinv*( Bldir*Vpast(1) - 2*Rleft*Ipast(1) + Vleft(n) +Vleft(n+1));
    Vnext(Nl+1) = B2inv*( B2dir*Vpast(Nl+1) + 2*Rright*Ipast(Nl) + Vright(n)+Vright(n+1));

    Vnext(indv) = e1*Vpast(indv) - e2*(Ipast(indv+0) - Ipast(indv-1));
    Inext(indi) = h1*Ipast(indi) - h2*(Vnext(indi+1) - Vnext(indi-0));
    Vpast = Vnext; Ipast = Inext; n = n + 1;
%-----
    string = strcat(num2str(1e9*t(n)), ' ns');
    x = [min(lV) max(lV)]; y = [1 1];
    plot(lV, Vnext, 'r', 'LineWidth', 2); grid on;
    line(x, y, 'LineWidth', 2); line(x, -y, 'LineWidth', 2);
    axis([min(lV), max(lV) -3 +3]); title (strcat('Line voltage (V) at t=',string));
    xlabel('distance, m'); ylabel('V, V'); drawnow;
end
```

6. Example of using 1D Maxwell's equations

One immediate example of practical importance is using the 1D numerical solution for a wall (or shell) penetration study, at normal incidence of a plane wave. In order to do so, the FDTD algorithm in example 1 is simply modified by introducing an inhomogeneous dielectric layer of a certain thickness, d . None of the equations has to be changed; only the dielectric constant has to be made variable.

One additional step is the inclusion of medium conductivity; this step is accomplishing by rewriting Eqs. (7) in the form

$$\begin{aligned}\varepsilon \frac{\partial E}{\partial t} + \frac{\partial H}{\partial y} - \sigma E &= 0 \\ \mu \frac{\partial H}{\partial t} + \frac{\partial E}{\partial y} - \sigma^* H &= 0\end{aligned}\quad (19)$$

and applying the finite differences again. Here, σ, σ^* are the electric and magnetic conductivities (electric and magnetic losses), both having units of S/m. Eqs. (9) are slightly modified to

$$\begin{aligned}\varepsilon \frac{\partial E}{\partial t} + \frac{\partial H}{\partial y} - \sigma E &= 0 \Rightarrow \varepsilon \frac{E_k^{n+1} - E_k^n}{\Delta t} + \frac{H_{k+1/2}^{n+1/2} - H_{k-1/2}^{n+1/2}}{\Delta y} - \sigma \frac{E_k^{n+1} + E_k^n}{2} = 0 \\ \mu \frac{\partial H}{\partial t} + \frac{\partial E}{\partial y} - \sigma^* H &= 0 \Rightarrow \mu \frac{H_{k+1/2}^{n+3/2} - H_{k+1/2}^{n+1/2}}{\Delta t} + \frac{E_{k+1}^{n+1} - E_k^{n+1}}{\Delta y} - \sigma^* \frac{H_{k+1/2}^{n+3/2} + H_{k+1/2}^{n+1/2}}{2} = 0\end{aligned}\quad (20)$$

where from E_k^{n+1} and $H_{k+1/2}^{n+3/2}$ are again expressed in terms of the past values.

Example 3: Compile a MATLAB script that solves the problem of plane wave penetration through a 30cm ($\sim 1'$) thick brick wall with the relative dielectric constant of 7 and the medium conductivity of 0.02 S/m. The operation frequency is 1 GHz.

Solution: We modify the script to example 1 following Eqs. (20) and obtain

```
% EM data
epsilon = 8.85418782e-012; % vacuum, F/m
mu      = 1.25663706e-006; % vacuum, H/m
c       = 1/sqrt(epsilon*mu); % vacuum, m/s
eta     = sqrt(mu/epsilon); % vacuum, Ohm
%-----
% Trace data
D       = 600.0e-3; % Trace length, m
N1      = 300; dl = D/N1; % Number of spatial cells and spatial step
lE      = [0:dl:D]; % Y-coordinates for the E-field nodes, N1+1 total
lH      = [dl/2:dl:D-dl/2]; % Y-coordinates for the H-field nodes, N1 total
%-----
```

Example 3 (cont).

```

% Dielectric wall
w1 = Nl/4; w2 = 3*Nl/4; % indexes into wall coordinates
eps_r = 25; % relative dielectric constant
sigma = 0.02; % medium conductivity, S/m
eps_profile = epsilon*ones(size(lE)); % plain dielectric profile
sig_profile = zeros(size(lE)); % plain conductivity profile
eps_profile([w1:w2]) = eps_r*epsilon; % modify the profile
sig_profile([w1:w2]) = sigma; % modify the profile
%-----
dt = dl/c; % magic time step
T = 10e-9; % observation time, s
NT = round(T/dt); t = [0: dt: NT*dt]; % Number of time steps and time vector
delta = dt/dl;
%-----
% FDTD - marching on in time; initialization
Hnext = zeros(1, Nl+0); % fourth layer in the series of four (n+3/2)
Enext = zeros(1, Nl+1); % third layer in the series of four (n+1)
Hpast = zeros(1, Nl+0); % second layer in the series of four (n+1/2)
Epast = zeros(1, Nl+1); % first layer in the series of four (n+0)
indv = 2:Nl; % batch process in space (voltage or electric field)
indi = 1:Nl; % batch process in space (current or magnetic field)
%-----
% Electric field source and impedance boundary conditions
f = 1e9; % Frequency of interest
Eleft = 2*sin(2*pi*f*t); % Electric field source, left
Eright = 0*sin(2*pi*f*t); % Electric field source, right
Rleft = eta; % Surface impedance, left boundary
Rright = eta; % Surface impedance, right boundary

Blinv = 1/(Rleft*epsilon/delta + 1); % BC coefficient -left
Bldir = (Rleft*epsilon/delta - 1); % BC coefficient -left
B2inv = 1/(Rright*epsilon/delta + 1); % BC coefficient -right
B2dir = (Rright*epsilon/delta - 1); % BC coefficient -right

e1 = (1 - dt*sig_profile./(2*eps_profile))./...
(1 + dt*sig_profile./(2*eps_profile));
e2 = (dt./(dl*eps_profile))./...
(1 + dt*sig_profile./(2*eps_profile));
h1 = 1;
h2 = dt/(dl*mu);

% Main loop - "bootstrapping"
n = 1;
while n <= NT-1
    Enext(1) = Blinv*( Bldir*Epast(1) - 2*Rleft*Hpast(1) + Eleft(n) +Eleft(n+1));
    Enext(Nl+1)= B2inv*( B2dir*Epast(Nl+1)+ 2*Rright*Hpast(Nl)+ Eright(n)+Eright(n+1));

    Enext(indv) = e1(indv).*Epast(indv) - e2(indv).*(Hpast(indv+0) - Hpast(indv-1));
    Hnext(indi) = h1* Hpast(indi) - h2* (Enext(indi+1) - Enext(indi-0));
    Epast = Enext; Hpast = Hnext; n = n + 1;
    %-----
    string = strcat(num2str(1e9*t(n)), ' ns');
    x = [min(lE) max(lE)]; y = [1 1];
    plot(lE, Enext, 'r', 'LineWidth', 2); grid on;
    line(x, y, 'LineWidth', 2); line(x, -y, 'LineWidth', 2);
    axis([min(lE), max(lE) -3 +3]); title (strcat('Electric field (V/m) at t=',string));
    xlabel('distance, m'); ylabel('E, V/m');
    patch([lE(w1) lE(w1) lE(w2) lE(w2)],[-3 +3 +3 -3],'g','FaceAlpha', 0.1);drawnow;
    %-----
end

```

The corresponding plot to example 3 is shown in Fig. 9 that follows.

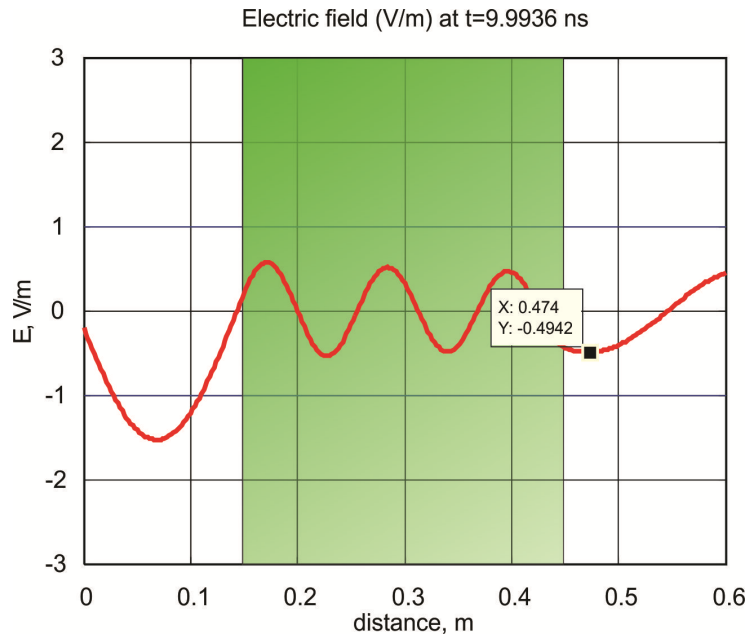


Fig. 9. Propagation of a normally incident plane wave through a brick wall. The source is on the left. The wave amplitude on the left is not that of the source (1V/m), due to reflection from the front dielectric boundary.

Using the numerical data one could estimate *the power transmission coefficient, T* , for normal incidence. The power per unit area transmitted by a plane wave (*Poynting vector*) is given by

$$P = \frac{E_a^2}{2\eta} \quad (21)$$

where E_a is the amplitude of the electric field. The ratio of the transmitted power to the power of the incident plane wave gives us the transmission coefficient in the form:

$$T = \frac{E_{a,\text{trans}}^2}{E_{a,\text{inc}}^2} \quad (22)$$

According to Fig. 9, $T = 0.494^2 = 0.244$ or approximately -6dB. In other words, about 75% of incident power is not transmitted through.

Other cases of practical interest many be considered in the same way.

7. 3D Maxwell's equations

7.1. Staggered grid

The finite differences given by Eq. (9) or by Eq. (20) are straightforwardly extended to a 3 D case, using the same idea of the staggered grid in three dimensions – see Fig. 10. The boundary conditions and the source are introduced in a similar way. The electric field is given at the edge centers. In particular, E_x is placed at the centers of the edges *parallel* to the x-axis, E_y is placed at the centers of the edges *parallel* to the y-axis, etc. The magnetic field is given at the face centers. In particular, H_x is placed at the centers of the faces *perpendicular* to the x-axis, H_y is placed at the centers of the faces *perpendicular* to the y-axis, etc.

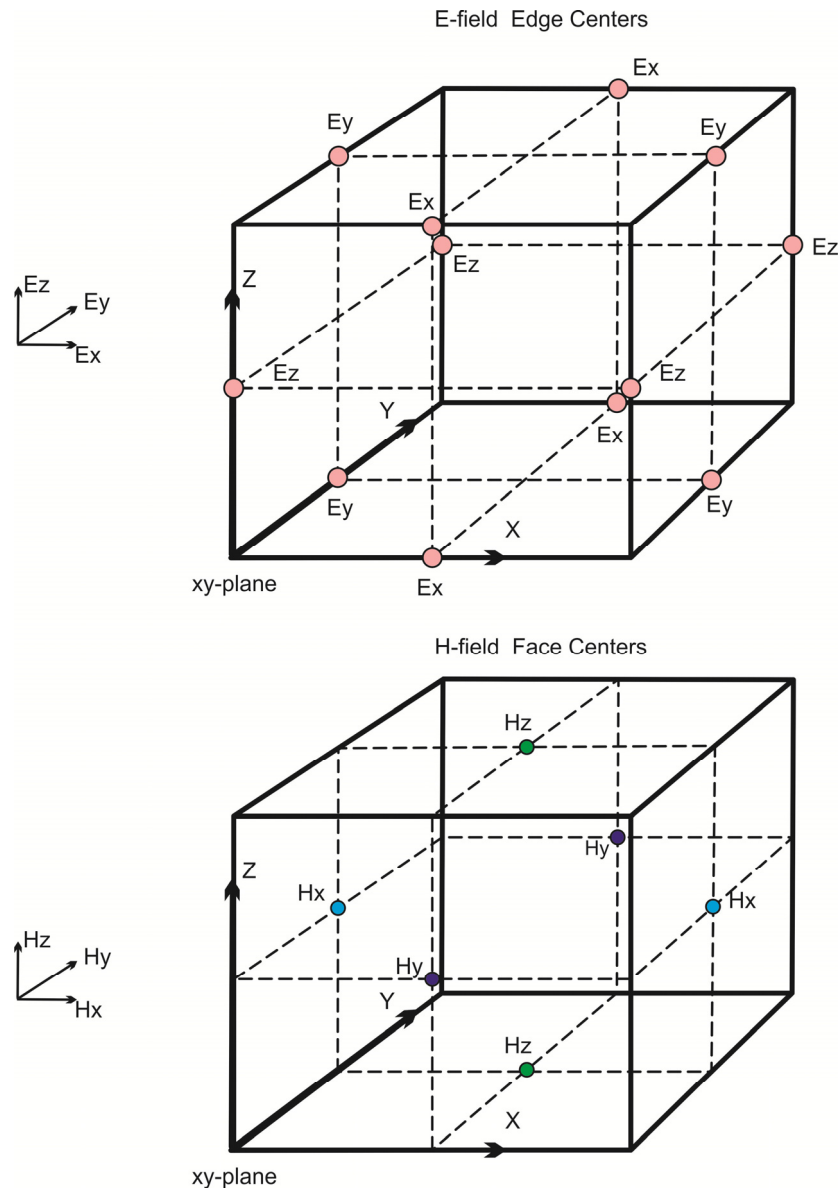


Fig. 10. Staggered grid in 3D.

Question: The 3D FDTD method seems to be complicated, especially in view of Fig. 10. Is this really the case? How is it programmed?

Answer: In fact, the FDTD equations are not really complicated. The above code shows the 3D FDTD marching on in time equations for a homogeneous medium:

```
% E-update (everywhere except on boundary; (40% of time))
ExN(:,2:Ny,2:Nz) = e1*Exp(:,2:Ny,2:Nz) + e2*(diff(HzP(:,2:Nz),1,2) -
diff(HyP(:,2:Ny,:),1,3));
EyN(2:Nx,:,2:Nz) = e1*Eyp(2:Nx,:,2:Nz) + e2*(diff(HxP(2:Nx,:,2:Nz),1,3) -
diff(HzP(:,2:Nz),1,1));
EzN(2:Nx,2:Ny,:) = e1*Ezp(2:Nx,2:Ny,:) + e2*(diff(HyP(:,2:Ny,:),1,1) -
diff(HxP(2:Nx,:,2:Nz),1,2));

% H-update (40% of time)
HxN = h1*HxP + h2*(diff(EyN,1,3) - diff(EzN,1,2));
HyN = h1*HyP + h2*(diff(EzN,1,1) - diff(ExN,1,3));
HzN = h1*HzP + h2*(diff(ExN,1,2) - diff(EyN,1,1));
```

7.2. Antenna feed (port) model

The (lumped) port model for a generator dipole antenna in FDTD⁸ is shown in Fig. 11 that follows. It occupies one unit cell of the FDTD grid.

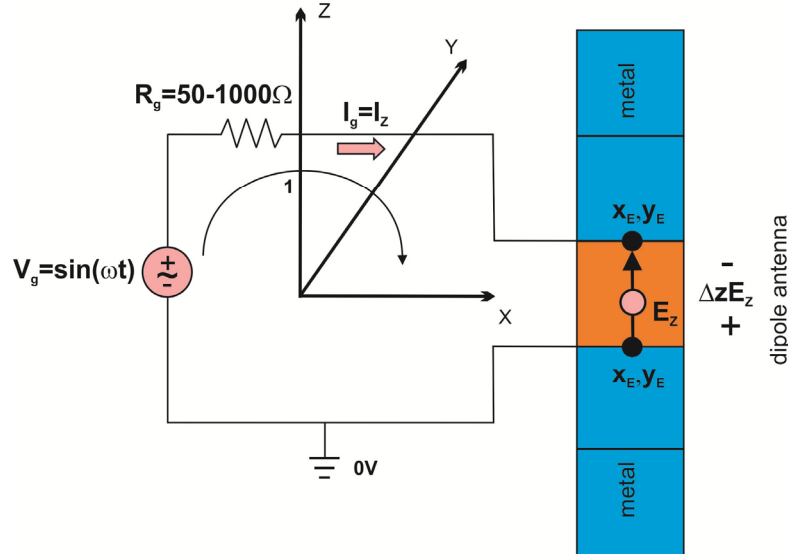


Fig. 11. FDTD port (antenna feed) model.

⁸ M. Picket-May, A. Taflov, and J. Baron, "FD-TD modeling of digital signal propagation in 3-D circuits with passive and active loads," *IEEE Trans. Microwave Theory and Techniques*, vol. 42, no. 8, Aug. 1994, pp. 1514 - 1523.

The generator circuit includes the open gap (antenna feed) with an electric field $E_z(t, x_e, y_e, z_e)$ that is updated from Maxwell's equations (1) in free space. We apply KVL to loop I as shown in the figure. This gives:

$$-V_g(t) + R_g I_g(t) - \Delta z E_z(t, x_e, y_e, z_e) = 0 \quad (23)$$

$$I_g(t) = \frac{1}{R_g} V_g(t) + \frac{\Delta z}{R_g} E_z(t, x_e, y_e, z_e) \quad (24)$$

According to the reference cited above, this is the “semi-implicit formulation” for the conduction current in the sense that this current relies in part upon the updated electric field to be determined as a result of the time stepping, and yet does not result in a system of simultaneous equations. This results in a numerically stable algorithm for arbitrary positive resistance values. Using the Ampere's law with an impressed current source from Eq. (1a) one could obtain the fully explicit formulation and thus find the generator current from Eq. (24).

For the receiver antenna, the same equivalent circuit as that shown in Fig. 11 holds, but with the voltage source set to zero. Eqs. (23)–(24) apply again; the voltage source is indeed zero. The receiver voltage (voltage across the load resistor) is then given by

$$V_L(t) = R_L I_L(t) \quad (25)$$

The transmitted and received powers, and the power transfer function are finally found in the same fashion as in Section II.

Example 4: A wireless link between two identical 15 cm long dipole blade antennas separated by 0.5m is studied using the 3D FDTD method at 900MHz. Every blade dipole has the width of 10mm. Establish value of the power transfer function between the TX circuit and the RX circuit using the definition Eq. (11c) of Section II. Both generator and load resistances are 50Ω .

Solution: We use MATLAB script `FDTD3D_1.m` to this lecture and obtain the script output shown in Fig. 12. The TX voltage source amplitude is 1V, the received voltage amplitude is 40 mV. Using the transfer function definition, Eq.(11c), of Section II one arrives at

$$T_P = \frac{P_{\text{load}}}{P_{\text{TX}}} = \frac{2R_g}{R_{\text{load}}} |\mathbf{T}_V|^2 = 2 \times (0.04)^2 = 0.0032 \quad (26)$$

or at -25dB power loss.

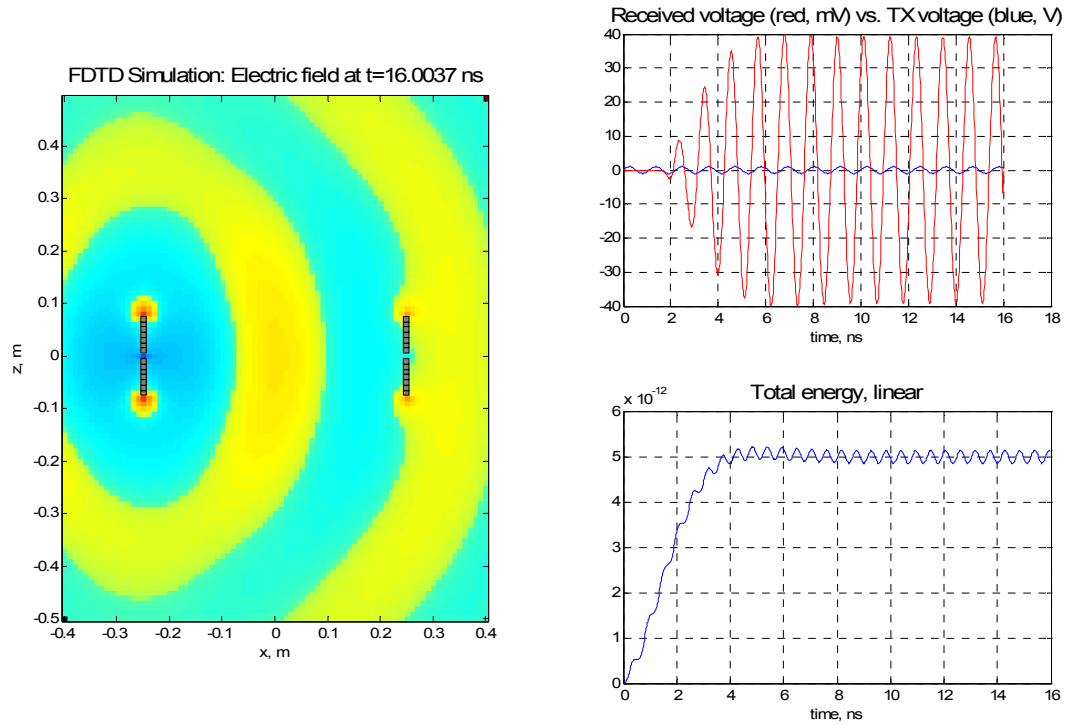


Fig. 12. Output to MATLAB script `FDTD3D_1.m`. A cross-section of a 3D space surrounding two blade dipole antennas is shown.

Review questions to Section III

1. How many independent Maxwell's equations do we know?
2. Why is Eq. (1a) called "Ampere's law"?
3. What is a simple solution to Maxwell's equations with zero sources?
4. What is the boundary condition on a surface of an ideal conductor (perfect metal)?
5. What is the boundary condition on a surface of a real metal?
6. Prove by direct substitution that Eqs. (4) give the solution to 1D Maxwell's equations (3).
7. List analogies between field and circuit parameters:

E-field –
H-field –
Dielectric permittivity –
Magnetic permeability –
8. Describe in your own words the concept of "marching on in time" for the FDTD method.
9. When you execute MATLAB script to example 1 (fdd1D_1.m), the field source is given by $E(t, y = 0) = 2 \sin \omega t$ [V/m]. On the other hand, you see on the screen a plane wave with the amplitude of 1V/m, but not of 2V/m as expected. Why is it so?
10. Using MATLAB script to example 2 (fdd1D_2.m) determine the amplitude of a standing voltage wave on the 600mm long transmission line when $R_g = 50\Omega$, $R_{\text{load}} = 1\Omega$ (impedance mismatch). All other line parameters and frequency remain the same.
11. Using MATLAB script to example 3 (fdd1D_3.m) determine the power transmission coefficient at normal incidence when the target is a human body medium with the relative dielectric constant of 50 and the conductivity of 0.5S/m. All other script parameters including medium thickness and frequency remain the same. *Hint:* Zoom in the MATLAB plot or change the scale in order to obtain sufficient resolution.

12. Using MATLAB script to example 4 (`fdtd3D_1.m`) determine the power transfer function between two 15 cm long blade dipoles separated by 0.3m. All other script parameters and frequency remain the same.

Problems

1. Maxwell's equations

Problem 1. Write down Maxwell's equations and explain the meaning of every equation in your own words.

Problem 2. Prove the vector identity $\nabla \cdot (\nabla \times \vec{A}) = 0$ for an arbitrary vector function $\vec{A}(\vec{r})$.

Problem 3. Prove the vector identity $\nabla \times (\nabla \phi) = 0$ for an arbitrary scalar function $\phi(\vec{r})$.

Problem 4. Prove the vector identity $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ for an arbitrary vector function $\vec{A}(\vec{r})$.

2. Boundary conditions

Problem 5.

1. List all boundary conditions on a material-material interface.
2. How is Eq. (2d) transformed for the interface between a perfect conductor and air?
3. How is Eq. (2b) transformed for the interface between a perfect conductor and air?

Problem 6. Formulate all boundary conditions on a dielectric-dielectric interface with $\epsilon_2 \neq \epsilon_1$ and $\mu_2 = \mu_1 = \mu_0$.

3. Analytical solution to Maxwell's equations in time domain

TBD

4. Numerical solution to Maxwell's equations in time domain

TBD

5. Equivalence of transmission line equations and 1D Maxwell's equations

TBD

6. Example of using 1D Maxwell's equations

Problem 7. Using MATLAB script `fdtd1D_problem7.m` (a slightly modified script to example 3) determine the power transmission coefficient at normal incidence when the target is a glass wall in a conference room having thickness of 12.5mm (approximately $\frac{1}{2}$ "), relative dielectric constant of 4.0 and the conductivity of 0.01S/m. The signal frequency is 2.45GHz. *Hint:* You have to carefully change all related script parameters and make sure that the script uses proper boundary conditions.

7. 3D Maxwell's equations

TBD