

Lecture IV

1D FDTD - coupled transmission lines

In this lecture, we continue to study the Finite-Difference Time-Domain (FDTD) method on staggered grid. We apply this method to two *coupled* terminated transmission lines with a realistic generator/load.

1. Concept

Consider two coupled transmission lines shown in Fig. 1. The line separation distance is D .

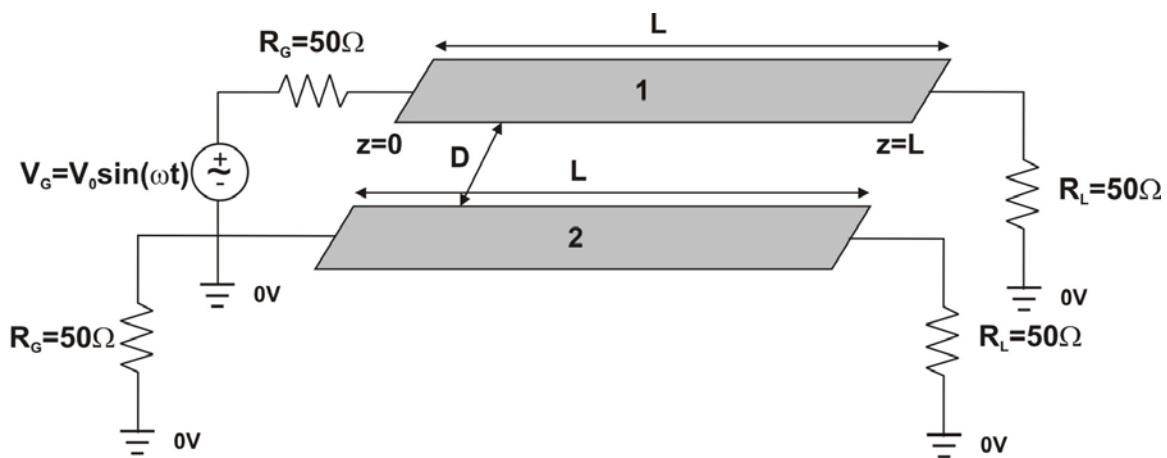


Fig. 1. A generic pair of two terminated coupled transmission lines. Left - line source; right - line termination.

What does it mean that lines are *coupled*? The answer is simple: there is a certain nonzero capacitance per unit length between the lines.

2. Short-circuit capacitances ("ground" capacitance matrix) of coupled transmission lines

In fact, there are two interrelated definitions of the capacitance matrix for multiconductor lines. With reference to Fig. 2a, we consider first the so-called Maxwellian capacitance matrix for three metal conductors. For arbitrary conductor voltages (or potentials) v_1, v_2, v_3 referenced to some *external* ground voltage and for arbitrary free charges on these conductors, Q_1, Q_2, Q_3 , one has

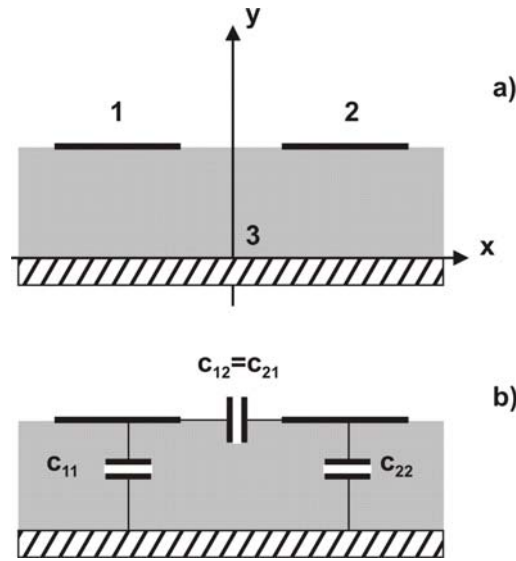


Fig. 2.a) - Conductor numbering and; b) - direct capacitances of the coupled microstrip line.

$$\begin{aligned}
 Q_1 &= C_{11}v_1 + C_{12}v_2 + C_{13}v_3 \\
 Q_2 &= C_{21}v_1 + C_{22}v_2 + C_{23}v_3 \\
 Q_3 &= C_{31}v_1 + C_{32}v_2 + C_{33}v_3
 \end{aligned} \tag{1}$$

The ij^{th} element of the Maxwell capacitance matrix is the total free charge on the i^{th} conductor when the potential of the j^{th} conductor is 1 V and *all* other conductors are grounded versus the external ground. Often the elements of the Maxwellian capacitance matrix are called *coefficients of capacitance*.

All coefficients of capacitance in Eq. (1) could be determined numerically, using the static MoM analysis to be studied in class. The capacitance matrix (1) is always symmetric, but its off-diagonal elements are negative. The capacitances in Eq. (1) are also called the *short-circuit* capacitances or capacitances to ground, as all the conductors, except for the one, are always grounded.

In practice, the reference voltage (or the reference potential) of the ground plane, v_3 , is set to zero volts, so that Eqs. (1) are simplified to

$$\begin{aligned}
 Q_1 &= C_{11}v_1 + C_{12}v_2 \\
 Q_2 &= C_{21}v_1 + C_{22}v_2
 \end{aligned} \tag{2}$$

3. Direct capacitances ("lumped" capacitance matrix)

One can further rewrite Eq. (2) in the form

$$Q_1 = [C_{11} + C_{12}](v_1 - 0) + [-C_{12}](v_1 - v_2) \quad (3)$$

$$Q_2 = [C_{21} + C_{22}](v_2 - 0) + [-C_{21}](v_2 - v_1)$$

or in the form

$$Q_1 = c_{11}(v_1 - 0) + c_{12}(v_1 - v_2) \quad (4)$$

$$Q_2 = c_{22}(v_2 - 0) + c_{21}(v_2 - v_1)$$

The elements c_{ij} ,

$$c_{ii} = \sum C_{ij}, \quad c_{ij} = -C_{ij} \quad (5)$$

are called in different sources as the *direct*, or *lumped*, or *self*- and *mutual* capacitances. They are always positive. Physically, c_{11} , is the capacitance of line 1 to ground when both line voltages are equal (self-capacitance); c_{12} is the capacitance *between* two lines (mutual capacitance) when the first line is grounded, etc. The lumped capacitances are shown in Fig. 2b.

4. Ground inductance matrix

The inductance matrix is called L_{ij} . According to Ref.

C. Wei, R. G. Harrington, J. R. Mautz, and T. K. Sarkar, "Multiconductor transmission lines in multilayered dielectric media," *IEEE Trans. Microwave Theory Tech.*, vol. 32, no. 4, April 1984, pp. 439-450.

the ij^{th} element of L is the magnetic flux passing between a unit length of the i^{th} conductor and the lower ground plane when one ampere of net z-directed electric current flows on the j^{th} conductor and there is no net z-directed electric current on any of the other conductors. Here, z is the coordinate perpendicular to the xy plane. It can be shown that

$$\hat{L} = \mu_0 \epsilon_0 [\hat{C}_0]^{-1} \quad (6)$$

where $\hat{C}_0$ is the ground capacitance matrix in vacuum of the same transmission line with *the dielectric being removed*. Both matrixes C_{ij} and L_{ij} are found using the static MoM analysis to be studied in class.

5. Telegrapher equations in the matrix form (Multiconductor TL or MTL equations)

Once we know these two matrixes (C_{ij} and L_{ij} ; both of them are real) from the MoM method to be studied in class, we can derive the time-domain transmission line equations (telegrapher equations) of a multiconductor line with N line voltages and N currents. Note that no other matrixes, e.g. the lumped capacitance matrix introduced above, become necessary. We introduce two vectors

$$\vec{v}(t, z) = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_N \end{bmatrix}, \quad \vec{i}(t, z) = \begin{bmatrix} i_1 \\ i_2 \\ \dots \\ i_N \end{bmatrix} \quad (7)$$

that describe voltages/currents on lines 1 through N and write the telegrapher equations in the form

$$\begin{aligned} \hat{L} \frac{\partial \vec{i}(t, z)}{\partial t} + \frac{\partial \vec{v}(t, z)}{\partial z} + \hat{R} \vec{i} &= 0 \\ \hat{C} \frac{\partial \vec{v}(t, z)}{\partial t} + \frac{\partial \vec{i}(t, z)}{\partial z} + \hat{G} \vec{v} &= 0 \end{aligned} \quad (8)$$

where $\hat{R}$ and $\hat{G}$ are resistance and conductance matrixes, per unit length, which are significant in the presence of material loss - see Lecture #2. Otherwise, they might be ignored. Similarly, a frequency-domain representation of telegrapher equations can be obtained.

In the lossless case, Eqs. (8) are reduced to

$$\begin{aligned} \frac{\partial \vec{v}(t, z)}{\partial t} &= -\hat{C}^{-1} \frac{\partial \vec{i}(t, z)}{\partial z} \\ \frac{\partial \vec{i}(t, z)}{\partial t} &= -\hat{L}^{-1} \frac{\partial \vec{v}(t, z)}{\partial z} \end{aligned} \quad (9)$$

The telegrapher equations (8) or (9) fully describe the coupled transmission lines including the so-called *differential* lines, with even or odd modes of propagation.

However, in order to find the response of the line terminated by certain networks, one has to consider these equations simultaneously with the equations describing the termination networks, i.e. consider the *boundary conditions* related to the MTL case.

6. Boundary conditions in the matrix form

The boundary conditions (Eqs. (1) of lecture#3) in the matrix form become

$$\begin{aligned}\vec{v}(t, z = 0) &= \vec{V}_G(t) - \hat{R}_G \vec{i}(t, z = 0) \\ \vec{v}(t, z = L) &= \hat{R}_L \vec{i}(t, z = L)\end{aligned}\quad (10)$$

Example 1: Formulate the boundary conditions to coupled TLs in Fig. 1 in the matrix form.

Solution:

$$\hat{R}_G = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}, \quad \hat{R}_L = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}, \quad \vec{V}_G = \begin{bmatrix} V_0 \sin \omega t \\ 0 \end{bmatrix} \quad (11)$$

7. MATLAB realization

The MATLAB realization of the MTL equations is conceptually straightforward. However, there are many little things to worry about.

1. We first convert the 1D arrays for voltage and current to 2D arrays. The first index becomes the line number, the second index still corresponds to the position in space.
2. Inductances and capacitances of the line are replaced by matrixes, so are the boundary conditions.
3. We plot voltages for all N lines.

Example 2 that follows present a code `mtl_03.m`. This is the code `tl_03.m` (the MATLAB code from example #2 of lecture #3), but modified for multiconductor transmission lines. Specifically, we consider the MTL line shown in Fig. 1.

The parameters of two (approximately $50 \, \Omega$ each) coupled microstrip lines on 62 mil FR4 with $\epsilon_r = 4.4$ are as follows:

- the line length is 300.0 mm;
- the line width is exactly 3mm;
- gap between lines is 4.2 mm.

The MoM method to be studied in class gives the following inductance and capacitance matrixes:

$$C = \begin{bmatrix} 122.2 \text{pF} & -4.1 \text{pF} \\ -4.1 \text{pF} & 122.2 \text{pF} \end{bmatrix}, \quad L = \begin{bmatrix} 0.3014 \mu\text{H} & 0.0216 \mu\text{H} \\ 0.0216 \mu\text{H} & 0.3014 \mu\text{H} \end{bmatrix} \quad (12)$$

Example 2 (no graphics included):

```

clear all;
%-----
D      = 300.0e-3;          % Trace length, m
Nz     = 400; dz = D/Nz;   % Number of spatial cells and spatial step
zkv    = [0:dz:D];         % Z-coordinates for the voltage nodes, Nz+1 total
zki    = [dz/2:dz:D-dz/2]; % Z-coordinates for the current nodes, Nz total
%-----
% Number of signal conductors
N      = 2;
% Two 3mm wide 500hm Tls on 62 mil FR4 separated by 4.2mm
% Direct ground capacitance matrix
C      = [1.222e-010 -0.041e-010;...
          -0.041e-010 1.222e-010];
% Direct ground inductance matrix
L      = [3.014e-007 0.216e-007;...
          0.216e-007 3.014e-007];
Cinv = inv(C);
Linv = inv(L);
%-----
cph    = 1/sqrt(L(1,1)*C(1,1)); % phase speed on the main line
dt     = 0.95*(1/cph)*dz;      % non-magic but safe time step
T      = 10e-9;                % observation time
NT     = round(T/dt); t = [0: dt: NT*dt]; % Number of time steps and time vector
%-----
% FDTD - marching on in time
% Initialization
Inext   = zeros(N, Nz+0); % fourth layer in the series of four (n+3/2)
Vnext   = zeros(N, Nz+1); % third layer in the series of four (n+1)
Ipast    = zeros(N, Nz+0); % second layer in the series of four (n+1/2)
Vpast    = zeros(N, Nz+1); % first layer in the series of four (n+0)
indv    = 2:Nz;           % batch process in space (voltage)
indi    = 1:Nz;           % batch process in space (current)
delta   = dt/dz;          % standard ratio
% Main loop - "bootstrapping"
n = 1;
%-----
% BCs
f = 1e9; Vg = 2*sin(2*pi*f*t); % Generator voltage
% Boundary conditions at z=0
RG = [50 0;...
      0 50];
% Boundary conditions at z=D
RL = [50 0;...
      0 50]; % experiment by changing them

B1inv = inv(RG*C/delta + eye(N)); % BC inverse -left
B1dir = (RG*C/delta - eye(N)); % BC direct -left
B2inv = inv(RL*C/delta + eye(N)); % BC inverse -right
B2dir = (RL*C/delta - eye(N)); % BC direct -right

n = 1;
while n <= NT-1
    VG = [Vg(n) + Vg(n+1); 0];
    VL = [0; 0];
    Vnext(:, 1) = B1inv*( B1dir*Vpast(:, 1) - 2*RG*Ipast(:, 1) + VG);
    Vnext(:, Nz+1) = B2inv*( B2dir*Vpast(:, Nz+1) + 2*RL*Ipast(:, Nz) + VL);

    Vnext(:, indv) = Vpast(:, indv) + delta*(-Cinv)*(Ipast(:, indv+0) - Ipast(:, indv-1));
    Inext(:, indi) = Ipast(:, indi) + delta*(-Linv)*(Vnext(:, indi+1) - Vnext(:, indi-0));
    Vpast = Vnext;
    Ipast = Inext;
    n = n + 1;
end

```

As a useful exercise, one may run the corresponding MATLAB script and observe the noise signal on a second line.

8. Magic time step - does it work?

The magic time step is now lost since the phase speed on one transmission line is affected by the presence of the second line and is not known a priori. This is why we use a smaller time step in example 2 above. When we try to use exactly the magic time found according to

$$\Delta t = \frac{\Delta z}{c_{ph}}, \quad c_{ph} = \frac{1}{\sqrt{C_{11}L_{11}}} \quad (13)$$

the method will diverge.

Note: By a subdivision of a MTL solution into the so-called *odd* and *even* modes of propagation, we are still able to overcome this difficulty and define the phase speeds for each mode separately. Then, the entire solution should be considered as a superposition of two modal solutions. The resulting approach becomes rather cumbersome though.

Such an approach quickly loses its sense when the lines with three and more signal conductors are considered.

Note: The MTL approach is applicable to a situation when there is any metal trace (or any metal object of the signal layer) close to the signal trace, not necessarily the second transmission line.

9. Bit (ramp) response and numerical dispersion

Now, similar to lecture #3, we consider a digital voltage signal in the form of a single bit as shown in Fig. 3.

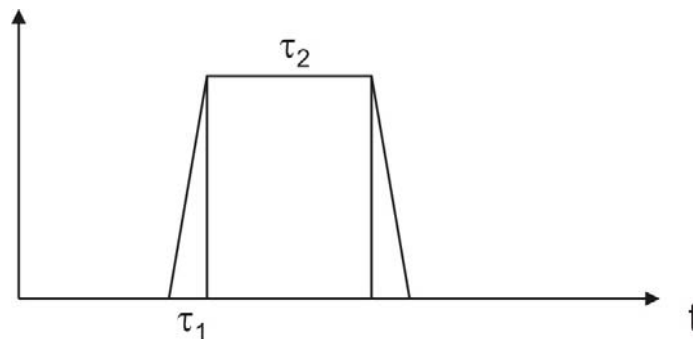


Fig. 4. Ramp excitation instead of the harmonic excitation. One single bit form is shown.

In Fig. 3, the ramp rise time is τ_1 , the flat top width is τ_2 . The MTL method of example 2 may be converted to the present case, the corresponding approach has been described in lecture #3 - see example 3 of lecture#3. The corresponding MTL MATLAB code for this lecture has the name `mtl_04.m`.

Note: Using the MATLAB scripts `mtl_04.m` for two coupled TLs specified in section 7 one can establish that

1. The ramp response provides a greater coupling as compared to the sinusoidal signal of the same amplitude (due to higher frequencies involved);
2. The bit form on the main line becomes significantly distorted too, due to the coupling effect. This fact may be proved by comparison with the solution obtained for (in contrast to Eq. (12))

$$C = \begin{bmatrix} 122.2\text{pF} & 0 \\ 0 & 122.2\text{pF} \end{bmatrix}, \quad L = \begin{bmatrix} 0.3014\mu\text{H} & 0 \\ 0 & 0.3014\mu\text{H} \end{bmatrix} \quad (14)$$

which implies no coupling, while keeping all other parameters the same.

10. How to calculate the MTL capacitance and inductance matrixes for transmission lines?

One way is to use an ANSOFT HFSS v. 12 FEM solver. However, the values obtained there may be even less accurate since the solver cannot handle the 2D geometry and always requires a 3D termination. Plus, setting up the corresponding Ansoft HFSS project will require certain skills. We shall acquire those skills in class.

Yet another way is to use a homemade MoM (actually FEM) software that is being introduced in the homework assignment to this lecture. This software calculates the static capacitance/inductance matrixes for various coupled transmission lines using an adaptive mesh refinement, similar to what Ansoft HFSS really does.

HW#4 exercise

This exercise is based on the MATLAB code `mtl_04.m` for one bit propagating along the 300mm long transmission line. We investigate the line coupling between two 50Ω microstrips when the gap between the lines increases to 9 mm.

1. Run a GUI script `struct2d.m` from your HW folder first. Change the geometry for two coupled microstrip lines on FR4 to

- gap between lines is 9.0 mm.

Keep all other line parameters the same. Save the mesh. Close the GUI script.

2. Run the script `main_static.m` from your HW folder next. This script will output the capacitance and inductance matrixes for two coupled TL lines at a wider spacing.
 3. Insert the new matrixes into the original FDTD script `mtl_04.m`.
 4. Observe the solution for the line voltage.
 5. Plot this solution at $t = 1.7\text{ns}$.
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Appendix A. Difference between SPICE and FDTD

A.1. How does the SPICE for TLs work?

To solve for a *lossless* transmission line, a SPICE circuit analysis program uses the method of *characteristics*. The method of characteristics provides an exact solution of the basic transmission line equations.

The method of characteristics is easily implemented in the 1D case of a lossless line, for the signal with one characteristic speed, and with either linear or nonlinear boundary conditions. Nonlinear TLs (the so-called NLTL) can be considered too.

The idea of this method is to follow any fixed value of voltage (e.g. 1V) in time, i.e. use the exact analytical solution for the two waves traveling along the line back and forth - see Fig. A1. These voltage values propagate along the line with a certain fixed phase speed, and are reflected from the boundaries.

Characteristics (paths of every particular voltage value in the zt -plane) are conventionally plotted as straight curves, which change their slope at the boundaries. For nonlinear lines, the phase speed may weakly increase with the voltage amplitude. Here, the effect of overturning (or pulse compression) is coming into path, which is characterized by the intersection of those characteristics.

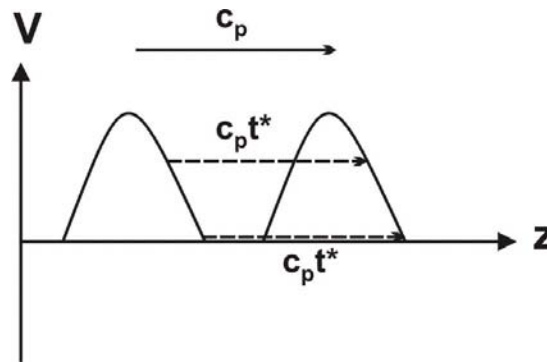


Fig. A1. Moving particular voltage values with the constant phase speed. In the zt -plane, paths of every particular voltage form straight lines - characteristics.

Mathematically, the solution for a lossless single-conductor line of length L with travel time $T = L/c_p$ is expressed only in terms of termination voltages and currents on the line; it has a form

$$V(0, t) = Z_0 I(0, t) + V_{\text{REF}}(L, t - T)$$

$$V_{\text{REF}}(L, t - T) = V(L, t - T) - Z_0 I(L, t - T)$$

$$V(L, t) = -Z_0 I(L, t) + V_{\text{INC}}(0, t - T) \quad (\text{A1})$$

$$V_{\text{INC}}(0, t - T) = V(0, t - T) + Z_0 I(0, t - T)$$

$$Z_0 = \sqrt{\frac{L}{C}}, \quad c_p = \frac{1}{\sqrt{LC}}$$

where $V_{\text{REF}}, V_{\text{INC}}$ are the reflected voltage signals from two opposite sides of the line. In the simplest case of a matched load, $R_L = Z_0$, there is no reflection from the load, so that from Eqs. (A1) one has the solution for the output voltage in the form of a time-delayed replica of the input voltage

$$2V(L, t) = 2V(0, t - T) \quad (\text{A2})$$

For a non-matched load, the solution becomes more involved, but it is also quite simple. Using Eqs. (A2), for any input voltage profile $V(0, t)$ one could recursively track the output voltage $V(L, t)$. The whole dynamics of the transmission line FDTD propagation is thus skipped. Other equivalent formulations are possible.

A2. Why does the SPICE use time domain?

Conventional analog SPICE-like circuit simulators, such as HSpice and Spectre are very powerful tools to predict the behavior of the circuits containing nonlinear devices. Once the overall circuit behavior is nonlinear, a time-domain description is more robust than the frequency-domain description.

A3. Why FDTD but not the method of characteristics?

The method of characteristics does not work or requires significant modifications

- i. in a lossy case;
- ii. for non-uniform (non-ideal) TLs or tapered TLs;
- iii. for non-linear TLs.

What is most important however, the method of characteristics does not work in a 2D or a 3D case. Otherwise, the SPICE circuit simulator will outperform the present FDTD solver in both speed and accuracy.

Ref: X. Qinwei, Li Zheng-Fan, P. Mazumder, and M. Jun-Fa, "Time-domain modeling of high-speed interconnects by modified method of characteristics," *IEEE Trans. Microwave Theory Techniques*, vol. 48, no. 2, Feb. 2000, pp. 323-327.