

Lecture VI

MoM I

In this lecture, we start to study the Method of Moments (MoM). We apply this method to the quasi-static problems first.

Part I Theory

1. Maxwell's equations

Consider a medium with electric permittivity ϵ and magnetic permeability μ and with no (fictitious) magnetic current density. The sources are given by (volumetric) electric current density J with the units of A/m^2 and by volumetric electric charge density ρ with the units of C/m^3 . Maxwell's equations for electric field intensity $\vec{E}$ [V/m] and magnetic field intensity $\vec{H}$ [A/m] have the form

$$\text{Gauss' law for electric field} \quad \epsilon \nabla \cdot \vec{E} = \rho \quad (1a)$$

$$\text{Faraday's law} \quad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad (1b)$$

$$\text{Gauss' law for magnetic field (no magnetic charges)} \quad \nabla \cdot \vec{H} = 0 \quad (1c)$$

$$\text{Ampere's law modified by displacement currents} \quad \nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} + \vec{J} \quad (1d)$$

2. How is the electrostatic approximation obtained?

It will be shown later in this lecture that the transverse electric fields of a **TEM mode** (the fields are in the xy -plane whilst the wave propagates along the z -axis) are curl-free (have the electric potential) and satisfy **static** Laplace's equation in a space domain with no charges (textbook, p. 153). A similar observation is valid for the transverse magnetic fields. When the charges are present (they usually reside on metal conductors) the Laplace's equation for the corresponding electric potential ϕ is transformed to the well-known **static** Poisson's equation.

The majority of transmission lines are those with the TEM (transverse electromagnetic) or quasi TEM fundamental mode, where both electric and magnetic fields are *perpendicular* (or "almost" perpendicular) to the direction of propagation. The representative transmission line cross-section (a microstrip) is shown in Fig. 1.

Propagation in this mode exists at any frequency, although above a certain frequency, higher modes may also exist. Examples of TEM transmission lines include the coaxial line, the microstrip, stripline, CPW, etc. The non-TEM transmission lines are hollow circular or rectangular metal waveguides.

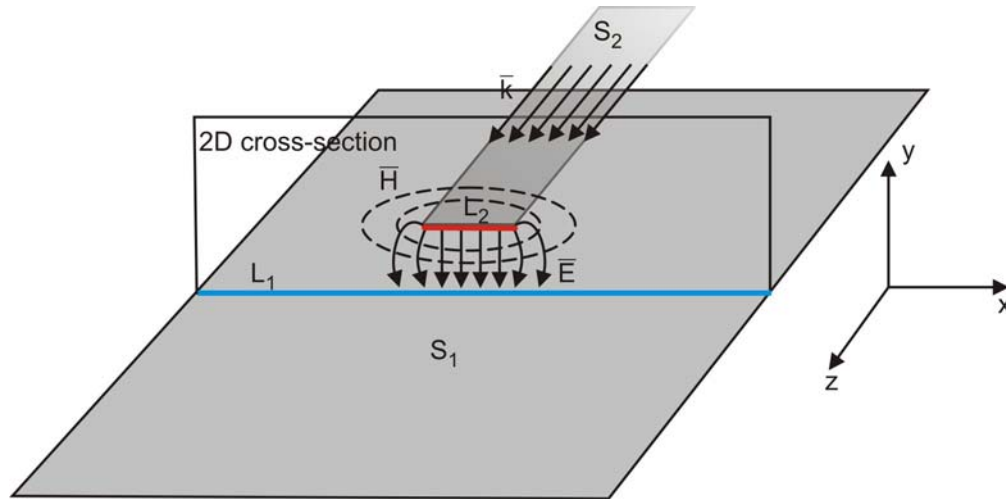


Fig. 1. A sketch of a transmission line cross-section. Electric and magnetic field are perpendicular to the direction of propagation of a signal on the line.

3. The electrostatic approximation

Let us now simplify Maxwell's equations (1) according to the assumption of a TEM field. From Faraday's induction law given by Eq. (1b) it follows that

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \Rightarrow (\nabla \times \vec{E})_z = \frac{\partial E_y(x, y)}{\partial x} - \frac{\partial E_x(x, y)}{\partial y} = 0 \quad (2)$$

In other words, the electric field in the xy -plane has zero curl. Thus, it is expressed as a gradient of a scalar function - the electric potential φ - in the form

$$\vec{E}(x, y) = -\nabla \varphi(x, y) \quad (3)$$

Next, we plug Eq. (3) into Gauss' law for electric field, Eq. (1a) and obtain

$$\epsilon \nabla \cdot \vec{E} = \rho \Rightarrow \Delta \varphi = -\frac{\rho}{\epsilon}, \quad \Delta = \nabla^2 \quad (4)$$

One can recognize in Eq. (4) a familiar electrostatic Poisson equation. For a predefined potential difference between the signal conductor and the ground plane, this equation will allow us to find the required charge distribution, ρ . Knowing the charges, it is straightforward to find static capacitance and the characteristic impedance of the line. Amazingly enough, the dynamic transmission line problem is thus reduced to a purely *electrostatic* problem. All fields and charges are indeed functions of time, this dependence is however *hidden*.

4. Electrostatics or magnetostatics?

Let us apply a similar derivation for the TEM field to the Ampere's law given by Eq. (1d). We obtain

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \vec{J} \Rightarrow (\nabla \times \vec{H})_z = \vec{J}_z \quad (5)$$

The Gauss' law for magnetic field, $\nabla \cdot \vec{H} = 0$, in the xy -plane allows us to introduce the *magnetic vector potential* (it has only one component, parallel to a current on the line) in the form:

$$\vec{H}(x, y) = \frac{1}{\mu} \nabla \times \vec{A}, \quad \vec{A} = (0, 0, A_z) \quad (6)$$

From Eqs. (5), (6) we obtain

$$(\nabla \times \vec{H})_z = \frac{1}{\mu} (\nabla \times \nabla \times \vec{A})_z = -\nabla^2 A_z \Rightarrow \nabla^2 A_z = -\mu \vec{J}_z \quad (7)$$

Eq. (7) is a "competitor" to electrostatic solution (4); it allows us to find the current on the line from a given value of the vector potential on the line. In its turn, this solution leads to a static inductance of the line. Both static inductance and static capacitance are important for the complete solution. However, in practice, the second step (finding the inductance) is usually *omitted*, even though it may lead to a numerically superior algorithm. One apparent difficulty is related to operations with the vector potential, whose units (T/m) are hardly visualized compared to volts. Instead, we may find the static capacitance, C_0 , with a dielectric being removed. Since any TEM transmission line with no dielectric has the phase speed equal to that of free space, the inductance can then be found from the circuit relation in the form

$$c_0 = \frac{1}{\sqrt{LC_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad (8)$$

This inductance value is clearly not affected by the dielectric and will remain *the same* for the dielectric-filled line.

5. A 3D solution

A solution to Eq. (4) in free space ($\varepsilon = \varepsilon_0$) can be constructed by superposing the contributions from point charges in a small volume dV' ,

$$q = \rho dV' \quad (9)$$

at locations $\vec{r}'$:

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|} \quad (10)$$

This yields

$$\varphi(\vec{r}) = \int_V \frac{\rho(\vec{r}') dV'}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} \quad (11)$$

If the potential φ is known, Eq. (11) may be viewed as an *integral equation* for the charge density ρ . The integration kernel,

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} \quad (12)$$

is called *Green's function*; it satisfies the equation

$$\nabla_{\vec{r}}^2 G(\vec{r}, \vec{r}') = -\frac{\delta(\vec{r} - \vec{r}')}{\epsilon_0} \quad (13)$$

which is the Poisson's equation with the right-hand side in the form of a point charge (delta-function).

6. A 3D solution in terms of surface charge density

A more practical (transmission-line related) problem is that for a number of metal conductors, where charges only reside on their surfaces, but not in the volume. In this case, one should replace the volume integral in Eq. (11) by a surface integral and the volume charge density by a surface charge density. Furthermore, the electric potential φ is given on boundaries: for example, $\varphi = 0$ on the ground-plane conductor and $\varphi = 1V$ on the microstrip surface. We shall denote this given potential by φ_{spec} . Eq. (11) is then reduced to

$$\varphi_{\text{spec}}(\vec{r}) = \int_S \frac{\rho_s(\vec{r}') dS'}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} \quad (14)$$

which is again the integral equation for the surface charge density.

Note: The Method of Moments solves the **integral equation** - Eq. (14). It does not solve the original differential equation - Eq.(4).

Note: The Method of Moments automatically satisfies the boundary conditions - no fields at infinity. This is a huge advantage of the MoM over FEM and/or FDTD.

7. A 2D electrostatic solution for a transmission line

For an infinitely long transmission line, only one cross-section of the line is usually of interest, which is perpendicular to the z -axis. The field and charge distributions remain the same (to within a different phase) for any cross-section, as shown in Fig. 2. Therefore, we can work in terms of phasors (factor the multiplier $\exp(j\omega t)$ if necessary).

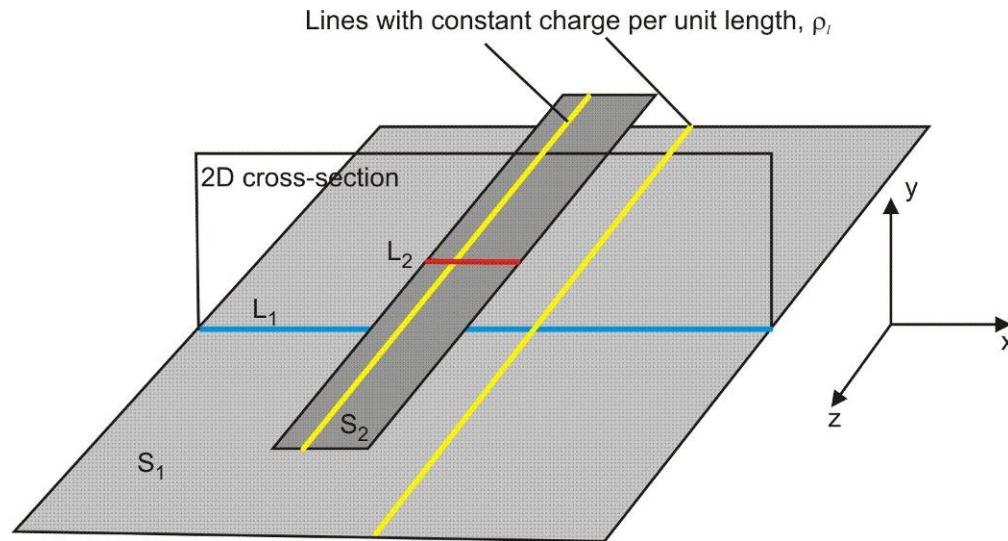


Fig. 2. Reduction of a transmission-line problem to a 2D electrostatic formulation.

This gives us a 2D electrostatic problem, which is *exactly* equivalent to the transverse TEM fields. With reference to Fig. 2, the surface integral in Eq. (14) over the metal surface $S = S_1 + S_2$ will be a **product of two line integrals**: one integral is along any line parallel to the z -axis, and another - over the cross-section contour $L = L_1 + L_2$.

There is no need to find the integral along a line parallel to the z -axis since we know, from electrostatics, that any line with a charge uniformly distributed along the line will create the electric potential

$$\varphi(\vec{r}) = -\frac{\rho_l \ln|\vec{r} - \vec{r}'|}{2\pi\epsilon_0} \quad (15)$$

where ρ_l is charge density per unit length. Then, the integral given by Eq. (14) is reduced to a contour integral

$$\varphi_{\text{spec}}(\vec{r}) = -\int_L \frac{\rho_l(\vec{r}') \ln|\vec{r} - \vec{r}'|}{2\pi\epsilon_0} dL' \quad (16)$$

With reference to Fig. 2, the *integration* variable, $\vec{r}'$, in Eq. (15) belongs to contour $L = L_1 + L_2$, whereas the *observation* variable, $\vec{r}$, belongs to the same contour too. $\varphi_{\text{spec}}(\vec{r})$ is zero on the ground-plane conductor (contour L_1) but equals 1 V on the microstrip surface ((contour L_2)). Eq. (16) is the base for the transmission-line modeling and capacitance calculation, per unit length.

8. MoM equations

Consider a situation when the entire contour L is divided into N small straight segments (edges), of equal or, maybe, non-equal lengths - see Fig. 2. The MoM introduced in 1960s, solves Eq. (16) in the following way: the unknown charge density is expanded into N *known* basis functions $s_n(\vec{r})$,

$$\rho_l(\vec{r}) = \sum_{n=1}^N a_n s_n(\vec{r}) \quad (17)$$

but with *unknown* coefficients a_n . In its simplest version, which is used here, a basis function $s_n(\vec{r})$ is just "one bit" - it is equal to one for edge n and equals zero for all other edges of the contour - see Fig. 3. In other words, we consider a piecewise-constant charge distribution with unknown coefficients - the staircase distribution.

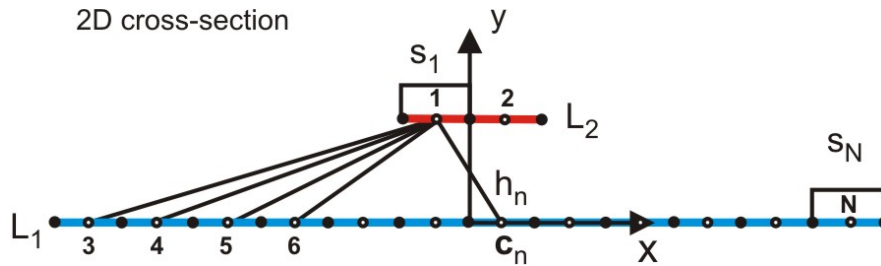


Fig. 3. The 2D cross-section from Fig. 2 and the (uniform) MoM edge subdivision. Edge centers are marked by white circles.

The first step of MoM is to substitute expansion (17) into Eq. (16) and interchange summation and integration. This gives

$$\varphi_{\text{spec}}(\vec{r}) = - \sum_{n=1}^N a_n \int_L \frac{s_n(\vec{r}') \ln|\vec{r} - \vec{r}'|}{2\pi\epsilon_0} dL' \quad (18)$$

Even if we would be able to find all resulting integrals analytically, this step will still not give us a complete system of equations for unknown coefficients a_n . To obtain this system, we additionally multiply both sides of Eq. (18) by every basis function,

$s_1(\vec{r}), s_2(\vec{r}), \dots, s_N(\vec{r})$, and then integrate the resulting expression one more time over the contour L . These steps give

$$\begin{aligned}
 V_1 &= \int_L \varphi_{\text{spec}}(\vec{r}) s_1(\vec{r}) dL = \sum_{n=1}^N a_n \left\{ - \iint_{L,L} \frac{s_1(\vec{r}) s_n(\vec{r}') \ln|\vec{r} - \vec{r}'|}{2\pi\epsilon_0} dL' dL \right\} = \sum_{n=1}^N a_n z_{1n} \\
 V_2 &= \int_L \varphi_{\text{spec}}(\vec{r}) s_2(\vec{r}) dL = \sum_{n=1}^N a_n \left\{ - \iint_{L,L} \frac{s_2(\vec{r}) s_n(\vec{r}') \ln|\vec{r} - \vec{r}'|}{2\pi\epsilon_0} dL' dL \right\} = \sum_{n=1}^N a_n z_{2n} \\
 &\dots \\
 V_N &= \int_L \varphi_{\text{spec}}(\vec{r}) s_N(\vec{r}) dL = \sum_{n=1}^N a_n \left\{ - \iint_{L,L} \frac{s_N(\vec{r}) s_n(\vec{r}') \ln|\vec{r} - \vec{r}'|}{2\pi\epsilon_0} dL' dL \right\} = \sum_{n=1}^N a_n z_{Nn}
 \end{aligned} \tag{19}$$

One can see that Eqs. (19) could be compactly written in the form of a system of linear equations:

$$\begin{bmatrix} V_1 \\ V_2 \\ \dots \\ V_N \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & \dots & z_{1N} \\ z_{21} & \dots & \dots & z_{2N} \\ \dots & \dots & \dots & \dots \\ z_{N1} & z_{N2} & \dots & z_{NN} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_N \end{bmatrix}, \quad z_{mn} = - \iint_{L,L} \frac{s_m(\vec{r}) s_n(\vec{r}') \ln|\vec{r} - \vec{r}'|}{2\pi\epsilon_0} dL' dL, \quad V_m = \int_L \varphi_{\text{spec}}(\vec{r}) s_m(\vec{r}) dL$$

(20)

for unknown coefficients a_n . The (often but not always) symmetric matrix Z is usually called the **MoM** (or **impedance**) matrix. Thus, the MoM method solves the system of linear equations (20).

The present MoM method is called the *Galerkin* method; it uses the special testing procedure (second integration) plus the same basis functions as testing functions.

In another, simpler MoM formulation (*collocation* or *point matching*), we just pick up N discrete versions of Eq. (18), say, for every edge midpoint - see Fig. 3 - and thus arrive at another complete system of N MoM equations, which may be very similar to the present one.

9. How to find the MoM integrals?

The elements of the MoM matrix are expressed in the form of double integrals; those integrals should be evaluated either numerically or analytically. Here, we prefer to use the simplest midpoint numerical integration, even though the analytical integration, or a more accurate numerical integration, are both possible for the static 2D or 3D problem. However, the more accurate integration formulas will not necessarily give us the more accurate final results since the initial error introduced by the staircase approximation itself *cannot be undone*, by any integration formulas.

Moreover, a "fine" integration may make this error even more profound, by underscoring the staircase basis function profile. On the other hand, the midpoint integration tries to "smooth" the staircase profile (it is not applied to the edges) and may surprisingly improve the solution accuracy compared to higher-order schemes.

Thus, the midpoint integration gives

$$z_{mn} \cong -\frac{h_m h_n \ln|\vec{c}_m - \vec{c}_n|}{2\pi\epsilon_0}, \quad V_m \cong h_m \varphi_{\text{spec}}(\vec{c}_m) \quad (21)$$

where $h_n, \vec{c}_n$ are edge length and midpoint edge position, respectively. Note that Eq. (21) predicts that our MoM matrix will be identical with that for the point matching, to within edge length multiplication. The multiplication by h_m in Eq. (21) may be cancelled for both z_{mn}, V_m .

10. Self-terms of the MoM matrix

The approximation given by Eq. (13) fails when $m = n$. In this case, we must use the analytical integration. Without the loss of generality, we assume that an edge under test is located along the x -axis, starting with $x=0$. This gives

$$z_{nn} = -\int_0^{h_n} \int_0^{h_n} \frac{\ln|x - x'|}{2\pi\epsilon_0} dx dx' = -\frac{h_n^2 (\ln h_n - 1.5)}{2\pi\epsilon_0} \quad (22)$$

if we use integration by parts. The accuracy of self-term calculations is critical for the MoM method when the number of unknowns (edges) is small. For a large number of unknowns, the importance of self-terms diminishes.

11. Potential of the reference conductor

It is customary to choose the potential of the reference conductor (ground plane) being equal to zero, so that

$$\varphi_{\text{spec}}(\vec{r} \in L_1) = 0 \quad (23)$$

in Fig.2 or Fig.3. All other conductors may have different nonzero potentials (positive or negative).

12. Charge conservation equation

The condition of total electric charge being equal to zero is not explicitly imposed into the above equations. Neither it is present in some textbooks including the class text. However, a simple numerical experiment with a transmission line having two non-equal conductors will show us that this condition will fail, even though such an effect may

initially have no visible influence on the static capacitance obtained by the MoM method - see below. Only in some special cases –two equal conductors or for a conductor above an infinite ground plane calculated by the method of images – the condition of the zero net charge will be satisfied automatically.

In order to satisfy this condition in every case, we will use the approach suggested in Ref. [M. B. Bazdar, A. R. Djordjevic, R. G. Harrington, and T. K. Sarkar, “Evaluation of quasi-static matrix parameters for multiconductor transmission lines using Galerkin’s method,” *IEEE Trans. Microwave Theory Tech.*, vol. 47, no. 7, July 1994, pp. 1223-1228.].

Namely, we divide each of equations (20) (or (21)) by the corresponding edge length, h_m , then subtract the last of Eqs. (20) (which corresponds to the last segment on the reference conductor) from the others, and finally replace this last equation by the charge conservation law, in the form

$$\begin{bmatrix} V_1 \\ V_2 \\ \dots \\ 0 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & \dots & z_{1N} \\ z_{21} & \dots & \dots & z_{2N} \\ \dots & \dots & \dots & \dots \\ h_1 & h_2 & \dots & h_N \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_N \end{bmatrix} \quad (24)$$

Part II Practice

1. MoM function mom2D

MATLAB function `mom2d.m`, the text of which is given below, implements and solves MoM Eqs. (20)-(24) given above for a metal transmission line (coaxial, microstrip, stripline, CPW, etc) in air. It can be modified in a number of ways in order to enable faster and more accurate computations for unknown MoM coefficients a_n . These coefficients have the meaning of charge density, per unit length, for a given edge; their product by edge length will give use total charge per edge, but still in C/m, since we consider line charges.

```
function [charge, a] = mom2d(xs, ys, xe, ye, phi);
%MOM2D Compute charge distribution for 2D electrostatics by MoM
% for an air-filled structure
%
% Syntax:
%   [charge, a] = mom2d(xs, ys, xe, ye, phi);
%
% Description:
% INPUTS:
%   xs      = x-coordinates for edge starting points (row)
%   ys      = y-coordinates for edge starting points (row)
%   xe      = x-coordinates for edge ending points (row)
%   ye      = y-coordinates for edge ending points (row)
```

```

% phi = potential for given edges (row)
% OUTPUTS:
% a = MoM coefficients a(n)
% charge = total line charge on each element (C/m)

const.epsilon = 8.85418782e-012; % ANSOFT HFSS value
const.mu = 1.25663706e-006; % ANSOFT HFSS value
const.c = 1/sqrt(const.epsilon*const.mu);
const.eta = sqrt(const.mu/const.epsilon);

x = 0.5*(xs + xe); % Observation/integration points
y = 0.5*(ys + ye); % Observation/integration points
h = sqrt((xe-xs).^2 + (ye-ys).^2); % Edge length

% Impedance matrix filling
N = length(x); % Number of edges
Z = zeros(N, N); % Impedance matrix
V = zeros(N, 1); % Right-hand side
Distance = zeros(1, N); % Distance vector
Kernel = zeros(1, N); % Integral kernel
eps = 1e-6*min(h); % Dummy variable

for m = 1:N
    Distance = sqrt((x-x(m)).^2 + (y-y(m)).^2) + eps;
    Kernel = log(Distance);
    Z(m,:) = -1/(2*pi*const.epsilon)*Kernel.*h(m);
    V(m) = phi(m)*h(m);
end
% MoM matrix (divide every row of MoM matrix and V(m) by h(m))
% Non-diagonal elements- single-point approximation
for m = 1:N
    Distance = sqrt((x-x(m)).^2 + (y-y(m)).^2) + eps;
    Kernel = log(Distance);
    Z(m,:) = -1/(2*pi*const.epsilon)*Kernel.*h;
    V(m) = phi(m);
end
% Diagonal (singular elements)
for m = 1:N
    Distance = sqrt((x-x(m)).^2 + (y-y(m)).^2) + eps;
    Kernel = log(Distance);
    Z(m,m) = -1/(2*pi*const.epsilon)...
        *h(m)*(log(h(m))-1.5);
end
% Charge conservation law:
% Subtract the last metal row from the others and replace the
% last metal edge equation by the conservation law
for m = 1:N-1
    Z(m,:) = Z(m,:) - Z(N,:);
    V(m) = V(m) - V(N);
end
% Last metal row
Z(N,:) = ones(1, N).*h;
V(N) = 0;
% Solution of linear equations
a = (Z\V)'; % MoM coefficients a(n) from matrix equation
charge = h.*a; % Total line charge per edge (C/m)

```

2. Static capacitance of a transmission line

Consider again a two-conductor transmission line with infinitely thin conductors shown in Fig. 2. For infinitely-thin metal-sheet conductors, MoM coefficients a_n have the

meaning of total charge density on *both* sides of the conductor *added* together. The total charge, per unit length, on the microstrip conductor (the upper conductor with the positive voltage) is thus

$$Q = \int_{L_2} \rho_l dL = \sum_{\text{Edge } k \text{ belongs to } L_2} a_k h_k \quad (25)$$

so that the capacitance per unit length of the air-filled transmission line is

$$C_0 = \frac{Q}{\Delta\varphi_{spec}} \quad (26)$$

Here, φ_{spec} is the predefined potential difference (or voltage) between two conductors. According to the above discussion, it makes sense to choose

$$\varphi_{spec}(L_1) = 0V, \varphi_{spec}(L_2) = 1.0V \quad (27)$$

so that Eq. (26) is reduced to

$$C_0 = \frac{Q}{1V} \quad (28)$$

where Q is the total charge (per unit length) on the microstrip (or upper) conductor.

3. Characteristic impedance of a transmission line with homogeneous dielectric

Consider a transmission line that is air-filled or filled with a homogeneous dielectric (coaxial line, stripline, embedded microstrip, etc.). In this case, the fundamental mode is the TEM mode, with the propagation speed

$$c_p = \frac{c_0}{\sqrt{\epsilon_r}} \quad (29)$$

where ϵ_r is the relative dielectric constant of the dielectric material. Simultaneously, the line characteristic impedance, Z_0 , becomes

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{\sqrt{\epsilon_r}} \sqrt{\frac{L}{C_0}} = \frac{1}{\sqrt{\epsilon_r}} Z_0^{\text{air}} = \frac{1}{c_0 \sqrt{\epsilon_r} C_0} \quad (30)$$

where

$$Z_0^{\text{air}} = \sqrt{\frac{L}{C_0}} = \frac{1}{c_0 C_0} \quad (31)$$

is the characteristic impedance of the transmission line in air.

4. Example – parallel-plate transmission line

The MATLAB script `impedance_problem1.m` in the MATLAB folder to this lecture uses the function `mom2.m` introduced above. It calculates the capacitance per unit length, and the characteristic impedance of an air-filled parallel-plate transmission line shown in Fig. 3, with *equal* width w and height d .

It turns out that our major difficulty will be in defining the proper geometry of the line, in MATLAB environment, and performing edge division. The script simultaneously outputs the exact value of static capacitance for the present problem numerically found in the class textbook, with a very large number of unknowns.

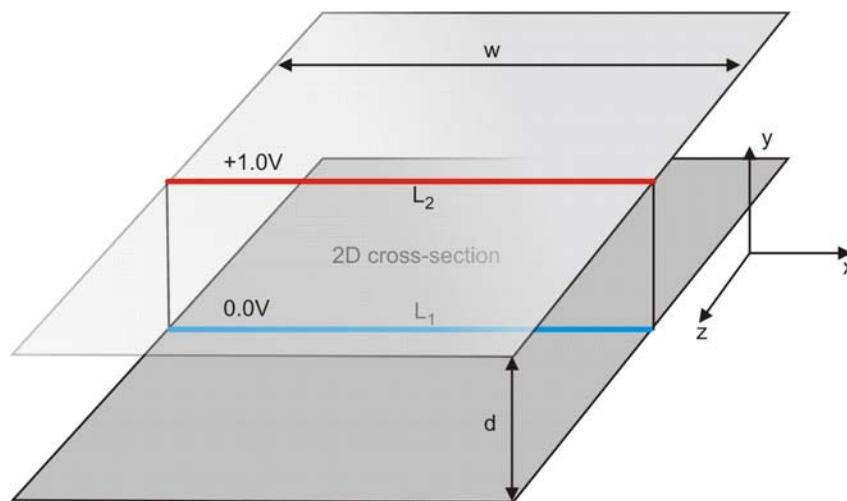


Fig. 3. Geometry for a parallel-plate transmission line.

```
%IMPEDANCE_PROBLEM1
%
% Syntax:
%   impedance_problem1
%
% Description:
%   Static capacitance and characteristic impedance of
%   a sheet parallel-plate TL.
%
const.epsilon    = 8.85418782e-012;           % ANSOFT HFSS value
const.mu         = 1.25663706e-006;           % ANSOFT HFSS value
const.c          = 1/sqrt(const.epsilon*const.mu);
const.eta        = sqrt(const.mu/const.epsilon);

d      = 0.01;           % Separation distance (m)
w      = 0.01;           % Plate width (m)
h      = w/20;           % Edge length (variable parameter)
eps_r  = 1;              % Dielectric constant (in the entire space)

% X- and Y-coordinates for starting and ending points
% Bottom plate
```

```

xs_b = [-w/2:h:w/2-h];
xe_b = xs_b + h;
ys_b = -d/2*ones(size(xs_b));
ye_b = ys_b;
% Add top plate
xs_t = [-w/2:h:w/2-h];
xe_t = xs_t + h;
ys_t = d/2*ones(size(xs_t));
ye_t = ys_t;
% Combine two meshes together
xs = [xs_b xs_t]; xe = [xe_b xe_t];
ys = [ys_b ys_t]; ye = [ye_b ye_t];
% Mesh size
N = length(xs)
% Potential for two conductors (V)
V(1:N/2) = 0.0; % ground plane
V(N/2+1:N) = 1.0; % top plate
% Solve the electrostatic problem (C, Z0)
[charge, a] = mom2d(xs, ys, xe, ye, V);
C0 = sum(charge(N/2+1:N)) % In F/m
Z0 = sqrt(eps_r)/(const.c*C0) % In Ohm
Error = abs(sum(charge))/max(abs(charge))
C_exact = 18.73e-12*sqrt(eps_r)
Z0_exact = sqrt(eps_r)/(const.c*C_exact)

```

The values of capacitance and characteristic impedance found for 40 edges (20 per plate) are

$$C = 18.78 \text{ pF/m}, \quad Z_0 = 177.6 \Omega \quad (32)$$

These values will indeed change a little when we increase the number of unknowns (number of edges).

HW#5 exercises

(textbook reading: pp. 153-170).

Problem 1. In the class textbook, an air-filled parallel-plate transmission line with $w=d$ was solved using the MoM method. However, all elements of the MoM matrix were computed using *exact* numerical integration, not the central-point approximation. As a result, the following convergence table was obtained depending on the *total* number of edges, N :

Table. 1 MoM convergence for the parallel-plate line from the class textbook (p. 164).

N	$C(\text{pF/m})$
20	18.37
50	18.59
100	18.66
200	18.70

- A. Fill out the corresponding table for the present MoM code (Section 4 of Part II).
- B. Assuming that the exact value of C is 18.73 pF/m, estimate, which code gives a smaller relative error for different N (converges faster). How could you explain this observation?
- C. How many unknowns are necessary for the present code to achieve a 1% accuracy in the static capacitance, C ?

Problem 2. Using the MoM method developed in this lecture find the approximate line capacitance and characteristic impedance for an **air-filled** finite-ground microstrip line shown in Fig. 4. The exact values for w and d are in fact not important, only their ratio counts for the final result. Use 100 segments for the active conductor and 100 segments for the ground plane conductor(s).

- A. Compare those results with the corresponding analytical model from the script `imp_microstrip.m`. This script uses formulas from Pozar's book. Is the difference significant?

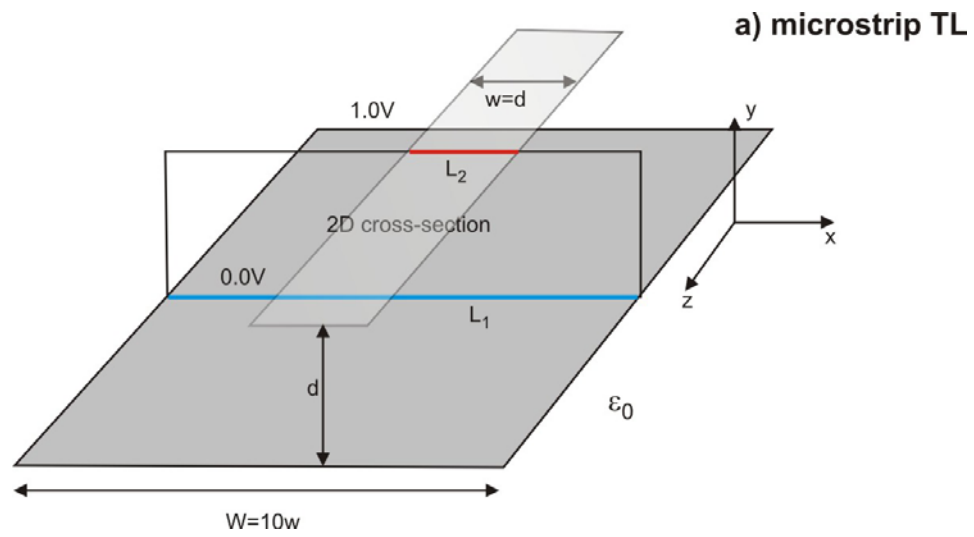


Fig. 4. Geometry for a microstrip transmission line with a finite ground plane.