

Lecture VIII

Basic Adaptive Mesh Refinement

1. Theory

1.1 A simple criterion for edge subdivision

To better account for a correct charge distribution we need to have more basis functions (fine mesh) in the domains with large charge density and less basis functions (coarse mesh) in the domains with small charge density since these domains will perhaps hardly contribute to capacitance anyway. According to Ref. [1], the goal of adaptive mesh refinement in electrostatics is to make sure that the total charge on every edge element stays the same. It means that we should have more elements in the areas with high charge density and less elements in almost neutral areas. In other words, the charge density multiplied with the edge length will give us the refinement criterion. This is shown schematically in Fig. 1 that follows

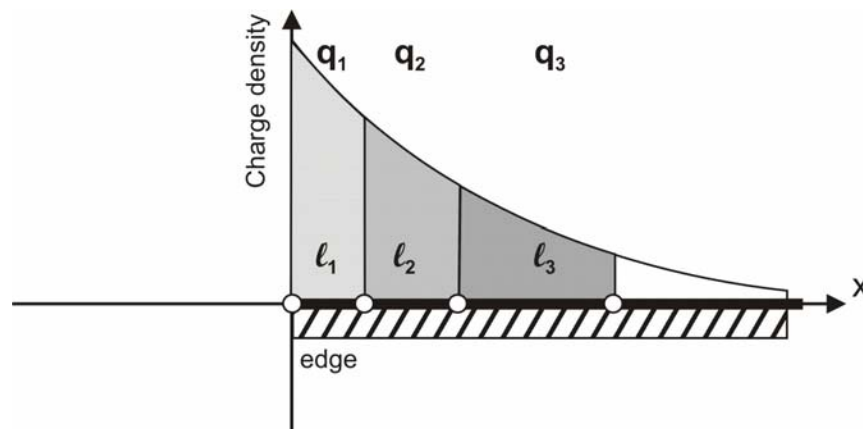


Fig. 1. Idea of adaptive mesh refinement. The mesh is "adapted" to solution behavior providing finer resolution for larger charge density - the area or total charge for every sub-edge stays approximately *the same*.

When a threshold is exceeded for the total edge charge, the edge is subdivided into 2 sub-edges, which decreases the total charge on every sub-edge by 50%.

1.2 More elaborate adaptive mesh refinement

The adaptive mesh refinement procedure attracted a lot of attention in the literature; many different ways to do so exist, which are related to different physical problems and numerical methods -see, for example, [2]-[6]. The fundamental question here is how to estimate the (largest) solution error and what domain to refine [4]. Above, we have intuitively chosen the domains with the largest charge densities as a potential "large-error" domain. This is generally true, but just imagine an opposite situation, when two

such oppositely charged domains are exactly canceled and there is another moderately charged domain that still gives the major contribution into static capacitance.

In FEM and magnetostatics, one could estimate the local solution error based on two complementary formulations, which solve the same problem twice, with two different approaches [4]. Having two independent solutions for the same mesh, we could estimate the largest local error between them, and then refine the mesh exactly in that domain. In MoM, which is based on integral equations but not on the differential equations, the error at any point is directly affected by all other calculated result, and the adaptive mesh refinement becomes more complicated [2]. However, some methods have been also proposed there - see, for example, [2], which are beyond the scope of the present basic analysis.

1.3 How it is done?

The total charge per edge is given by the array field `output.charge` of MATLAB structure `output`. After the initial MoM solution is obtained in the script `mom2d_md.m` (subfolder `codes`) we first find 25% of edges that have the highest total charge (highest areas):

```
charge      = abs(output.charge);
[dummy, index] = sort(charge);
index       = index(round(0.75*end):round(1.00*end));
             % 25 percent refinement per step
```

Then, we subdivide every such edge into two equal sub-edges. This procedure roughly halves the highest total charges and thus makes the total charges more uniform. After that, we update structure `mesh` and the voltage vector as shown in the script `mom2d_md.m`. This operation is simple since the same data on the surface normal and other edge parameters is used. We then run the MoM procedure again (the script `mom2d_md.m` has a loop versus the iteration number), find the new MoM solution, calculate a new error, and make another subdivision, etc.

1.4 A simple criterion of convergence

We need an integral quantity that will give us the criterion for the *relative* convergence or the *relative* error. Since we do not know the exact solution, the relative convergence is the only information that we may use with regard to the solution accuracy. Usually, an energy criterion is used, for the electric and magnetic fields. An appropriate substitute may be the static capacitance itself: we define the relative error at every mesh refinement step n as

$$\Delta S = \frac{C_n - C_{n-1}}{C_n} \quad (1)$$

We could refine the mesh multiple times. It is done in the script `mom2d_md.m`. As a result, we will have a relative solution error, which is a function of the iteration number. If this function tends to zero as the iteration number increases, the resulting solution is expected

to be reasonably accurate. Unfortunately, Eq. (1) does not work well for multiconductor lines, when multiple partial capacitances exist.

1.4 Power or energy convergence criterion

Instead, we will use the power (or energy) criterion. According to general EM, the power on the line is given by

$$\vec{P} = \frac{1}{2} \text{Re} \left(\int_{-\infty-\infty}^{\infty} \vec{E} \times \vec{H}^* dx dy \right) = \vec{z} \frac{1}{2} \text{Re}(V(z)I^*(z)) = \vec{z} \frac{1}{2} \frac{|V_0|^2}{Z_0} \quad (2)$$

This power could also be expressed through the surface current density, $\vec{J}_s$, in the form

$$\vec{P} = \vec{z} \frac{1}{2} \text{Re} \left(\int_{L_{\text{metal}}} V(l) J_s(l) dl \right) \quad (3)$$

where contour integration is made over all metal conductors. On the other hand,

$$\vec{J}_s = \vec{z} c_0 \rho_f \quad (4)$$

so that Eq. (3) gives an expression for the line power in terms of surface charges on metal conductors

$$\vec{P} = \vec{z} \frac{1}{2} c_0 \text{Re} \left(\int_{L_{\text{metal}}} V(l) \rho_f(l) dl \right) = \vec{z} \frac{1}{2} c_0 \text{Re} \left(\int_{L_{\text{metal}}} V(l) \varepsilon_r(l) \rho_{\text{total}}(l) dl \right) \quad (5)$$

Here, $\varepsilon_r(l)$ is the relative dielectric constant of a dielectric in contact with metal at a particular point, l , of the contour. Since we solely work with the real quantities, the real-part operation may be omitted everywhere in this text.

Eq. (5) is conveniently programmed using the existing solution and can be used to control the convergence. Instead of Eq. (1), one thus has

$$\Delta S = \frac{P_n - P_{n-1}}{P_n} \quad (6)$$

Note that the total electrostatic energy on the line (per unit its length) is given by

$$W = \frac{1}{2} \text{Re} \left(\int_{L_{\text{metal}}} V(l) \varepsilon_r(l) \rho_{\text{total}}(l) dl \right) \quad (7)$$

so that Eq. (6) will simultaneously give us either *energy* or *power* convergence criterion. This criterion is implemented in the MATLAB script `mom2d_md.m` (subfolder `codes`).

2. Microstrip example

2.1. Thin microstrip

The initial mesh may be coarse - we use the script `stuct2d.m` with the parameters shown in Fig. 2. The corresponding mesh is shown in Fig. 3.

AntLab/WPI Planar mesh generator

2D mesh generator (all domains (S1, S2, etc.) will be sorted in ascending order)

All units are millimeters

Name	xc	yc	Shape	Width	Height	isDiel	diel_r
S1	0	d+th1/2	☒ ☐	w	th1	☐	
S2	0	-th2/2	☒ ☐	W	th2	☐	
S3	0	d/2	☒ ☐	W	d	☒	4.4
S4	0	d/1.5	☒ ☐	1.2*W	2.5*d	☒	1
S5			☒ ☐			☐	
S6			☒ ☐			☐	
S7			☒ ☐			☐	
S8			☒ ☐			☐	
S9			☒ ☐			☐	
S10			☒ ☐			☐	

Set formula:

Name	Name	Name	Name	Name	Name	Name	Name
w	d	W	th1	th2			
Value(s)	Value(s)	Value(s)	Value(s)	Value(s)	Value(s)	Value(s)	Value(s)
1	1	10	0.05	0.15			

Triangle size

Fig. 2. The microstrip geometry. Note the triangle size value.

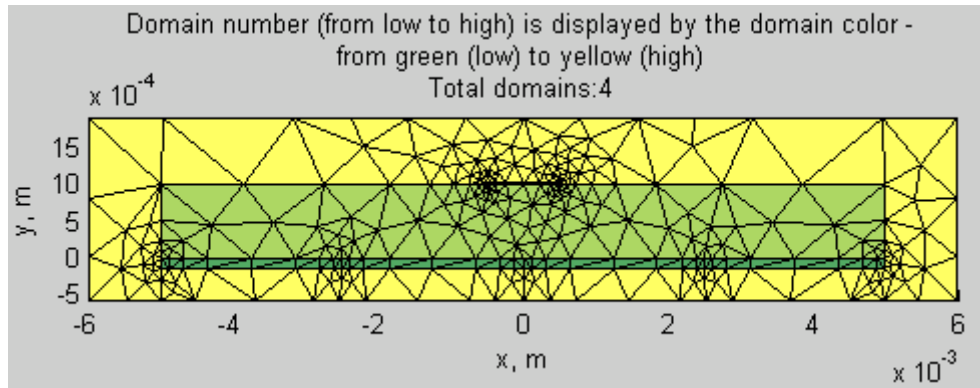


Fig. 3. The initial edge/triangle mesh.

We note that the present microstrip is now very thin - about 0.05 mm in thickness - much thinner than those considered in the past. A large error can be introduced into the solution by both the coarse mesh and, especially, by the central-point integration, which will fail for two very closely spaced edges.

The function `mom2d_md.m` is further used in the script `main.m`, with N adaptive passes ($N=10$ by default). It is controlled by the command

```
% Solve MoM equations and obtain TL parameters
N = 12; % number of adaptive mesh refinement steps
[output, mesh, V] = mom2d_md(mesh, V, N);
```

The script `main.m` has a convergence history (error in Eq. (1) multiplied by 100%) shown in Fig. 4.



Fig. 4. Relative error as a function of number of passes for 25% edge refinement criterion.

One can see that error behavior is not monotonic; similar non-monotonic behavior may be also observed in the commercial software, e.g. in Ansoft HFSS. When the number of edges to be refined becomes larger, the convergence curve generally may become more smooth. The final microstrip parameters, compared to the approximate analytical model for zero strip thickness [7], after 12th refinement step, are

```
Pozar =
      Z: 70.8215
  eps_eff: 3.1715

MoM =
      Z: 69.5863
  eps_eff: 3.1209
```

The agreement is indeed better than in Lecture #7 since we are considering thin microstrips and the corresponding analytical model becomes more accurate.

Below, in Table 1, we also give the absolute TL parameters (characteristic impedance) during the convergence process.

Table 1. Convergence of line impedance.

Iter. number N	1	2	3	4	5	6	7	8	9
Z0, Ω	88.9	68.9	69.8	69.7	69.5	69.6	69.6	69.6	69.6
No of Edges	110	138	173	217	272	341	427	535	670

Whilst the convergence with regard to static capacitance is satisfactory, the convergence with regard to other parameters might be worse. In particular, the effective dielectric constant converges slower than the capacitance. One can also observe from Table 1 how the total number of edges in the mesh increases with increasing the iteration number.

2.2. Thick microstrip

An experiment done with the thick microstrip (0.15 mm thickness) indicates that the mesh refinement process did not change significantly the initial result. One has

```
MoM =
      Z: 67.2168
  eps_eff: 3.0434
```

for the adaptive refinement process with $N=12$ used in this lecture. The corresponding characteristic impedance value was found for this thick microstrip using a two-port microstrip model in Ansoft HFSS. The Ansoft values averaged over low and moderate frequencies are relatively close to these values

```
FEM =
```

```

z: ~65.7
eps_eff: ~3.14

```

2.3. Another iteration process

The selection of edges given in section 1.2 may have a disadvantage that the edges near the corners are refined too fine, whilst other edges are virtually not refined. Namely, on the first iteration step, we select only corner edges to be refined. Adding another 25% of edges results in other corner edges. Since the solution is singular close to the corners, those new edges are likely again have maximum total charges, etc.

To reduce this effect we tried to refine not 25% of edges with the highest total charge but 25% of edges with the charge from 0% to 95% of highest. The corresponding changes are made in the script `mom2d_md.m`:

```

charge          = abs(output.charge);
[dummy, index]  = sort(charge);
index           = index(round(0.70*end):round(0.95*end));
                % 25 percent refinement per step

```

However, this iteration process resulted in a failure, despite the very good convergence curve behavior. We can see it by performing a simple numerical experiment. It is suspected that one reason can be the presence of very large and very small edges as neighbors.

References

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- [2] Dong-Kuk Park, Chang-Hoi Ahn, Soo-Young Lee, and Jung-Woong Ra, "Adaptive mesh refinement for boundary element method and its application to stripline analysis," *IEEE Trans. on Magnetics*, vol. 26, no. 2, March 1990, pp. 783-786.
- [3] Z. Cendes, D. Shenton, and H. Shahnasser, "Magnetic field computation using Delaunay triangulation and complementary finite element methods," *IEEE Trans. on Magnetics*, vol. 19, no. 6, Nov. 1983, pp. 2551-2554.
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- [7] D. M. Pozar, *Microwave Engineering*, Wiley, New York, 2005, 3rd edition.