

and integrates to

$$I \approx lh\eta_1 \left(kl \frac{\hat{n} \cdot \hat{x}}{\hat{n} \cdot \hat{z}} \right) \eta_1 \left(kh \frac{\hat{n} \cdot \hat{y}}{\hat{n} \cdot \hat{z}} \right) e^{i2kz'} \quad (6)$$

where

$$\eta_1(x) = \begin{cases} \sin(x)/x, & x \neq 0 \\ 1, & x = 0. \end{cases} \quad (7)$$

The result in (6) differs from that in (3) by the insertion of the two factors that account for the inclination of the rectangular plate with respect to the viewer. It is these inclination factors that were not properly accounted for in [1].

We express this result in spherical coordinates so that we can compare it to the corresponding expression in [1] more easily. With the origin of coordinates translated to the point (x', y', z') , we write

$$\hat{n} = \hat{x} \sin \vartheta \cos \varphi + \hat{y} \sin \vartheta \sin \varphi + \hat{z} \cos \vartheta, \quad 0 \leq \vartheta < \pi/2, \quad 0 \leq \varphi < 2\pi \quad (8)$$

with ϑ measured from the positive z -axis toward the negative one and φ from the positive x -axis toward the positive y -axis. The strict inequality on ϑ is none other than the PO condition $\hat{n} \cdot \hat{z} > 0$. From this and (6) we get that

$$I \approx lh\eta_1(kl \tan \vartheta \cos \varphi) \eta_1(kh \tan \vartheta \sin \varphi) e^{i2kz'}. \quad (9)$$

We compare this result to (7) in [1]. If in (9) we set $\varphi = 0$ or $\varphi = \pi$, we get

$$I \approx lh\eta_1(kl \tan \vartheta) e^{i2kz'} \quad (10)$$

while, if we set $\varphi = \pi/2$ or $\varphi = 3\pi/2$, we get

$$I \approx lh\eta_1(kh \tan \vartheta) e^{i2kz'}. \quad (11)$$

These are identical to (7) in [1] provided the pixel's area factor lh is removed, the taper factor $\cos^n \vartheta$ is inserted, and the sinc function is defined according to (7) above [the original, and prevailing, definition of the sinc function is [4] $\text{sinc}(x) = \eta_1(\pi x)$]. Setting $\varphi = 0$ or $\varphi = \pi$ in our (8) means that we have a flat plate rotated through an angle ϑ about the y -axis. Similarly, setting $\varphi = \pi/2$ or $\varphi = 3\pi/2$ means that we have a plate rotated about the x -axis. Accordingly, formula (7) in [1] interprets surfaces at the pixel level as flat plates rotated either about the vertical (y -axis) or the horizontal (x -axis). It ignores all other orientations.

III. CONCLUSION

For the hardwired graphics approach to PO we have obtained a formula that we consider an improvement over that of Rius *et al* [1]. We did this by using not only z -buffer information at the pixel level but also information about the normal. In effect, we have approximated the visible scattering surface by flat panels, each tangent to the surface at the pixel center. Since each panel is defined only over one pixel, adjacent panels do not necessarily intersect one another. Thus, as with the one-term (z -buffer) approximation of the surface at the pixel level, the original surface is broken up into a number of disconnected flat panels. This may be unpalatable to some, but it is the best approximation of the surface for the given information.

APPENDIX

From one of the reviewers we have learned that *XPATCHI* [5] (Version 5.4, dated Feb. 22, 1993), a code that also takes advantage of computer graphics, employs an expression in agreement with ours. The reviewer mentions that this formula "greatly enhances the accuracy of the PO solution." We have also been in communication with Dr. Rius, one of the authors of [1], and he has informed us that he will present his views on this matter in a forthcoming submission to this journal.

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A Perfectly Matched Anisotropic Absorber for Use as an Absorbing Boundary Condition

Zachary S. Sacks, David M. Kingsland, Robert Lee, and Jin-Fa Lee

Abstract—An alternative formulation of the "perfectly matched layer" mesh truncation scheme is introduced. The present scheme is based on using a layer of diagonally anisotropic material to absorb outgoing waves from the computation domain. The material properties can be chosen such that the interface between the absorbing material and free space is reflection-less for all frequencies, polarizations, and angles of incidence. This approach does not involve a modification of Maxwell's equations and is easy to implement in codes that allow the use of anisotropic material properties.

I. INTRODUCTION

Recently, Berenger introduced a novel absorbing boundary condition for truncating two-dimensional finite difference-time domain (FDTD) meshes [1]. His "perfectly matched layer" (PML) technique is based on using a layer of lossy material to absorb outgoing radiation from the computation domain. In the PML layer, the Cartesian field components are split into two subcomponents (i.e., $H_x = H_{xy} + H_{xz}$). Another formulation of the PML was given by Chew and Weedon [2]. Their approach is based on introducing complex coordinate stretching variables. In both approaches, Maxwell's equations were modified to add additional degrees-of-freedom. The modifications allow the specification of a lossy material layer such that a planar interface between the PML material and free

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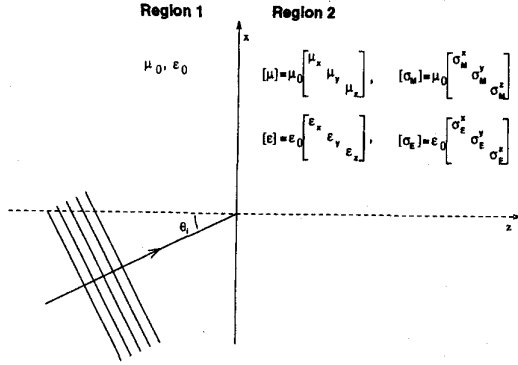


Fig. 1. A plane wave incident upon a half-space of diagonal anisotropic media.

space is reflectionless for all frequencies, polarizations, and angles of incidence.

In [1]–[3], the PML scheme was implemented for truncating regular FDTD grids. Berenger [1] and Katz *et al* [3] demonstrated that the PML technique offers much greater accuracy than conventional FDTD absorbing boundary conditions. Unfortunately, since Berenger's PML technique involves a modification of Maxwell's equations, its implementation requires significant modification of the standard FDTD time-stepping equations.

In this paper, an alternative approach for deriving a PML for mesh truncation is presented. This approach is based on using anisotropic material properties (ϵ , μ , σ_E , σ_M) to describe the absorbing layer. The material properties of the medium can be chosen such that a planar interface between the anisotropic medium and free space is perfectly reflectionless. In contrast to the conventional PML approach, the present approach does not require a modification of Maxwell's equations. Furthermore, implementation of the anisotropic PML approach is straightforward, especially in the frequency domain. In fact, implementing the current approach in an existing finite element method (FEM) code requires no modifications at all, provided that the code permits diagonally anisotropic materials.

A brief derivation of the anisotropic PML is presented below by considering the simple plane wave incidence problem shown in Fig. 1. First, the general solutions for $\vec{E}$ and $\vec{H}$ in the PML medium are derived. Next, the reflection coefficient at the interface is determined. Last, an example result is given to demonstrate the use of the anisotropic PML for FEM computations.

II. WAVES IN DIAGONALLY ANISOTROPIC MEDIA

The general time-harmonic form of Maxwell's equations are

$$\begin{aligned}\vec{\nabla} \cdot [\vec{\epsilon}] \vec{E} &= 0 \\ \vec{\nabla} \cdot [\vec{\mu}] \vec{H} &= 0 \\ \vec{\nabla} \times \vec{E} &= -j\omega[\mu] \vec{H} - [\sigma_M] \vec{H} \\ \vec{\nabla} \times \vec{H} &= j\omega[\epsilon] \vec{E} + [\sigma_E] \vec{E}.\end{aligned}\quad (1)$$

In this derivation, we consider materials where $[\vec{\epsilon}]$ and $[\vec{\mu}]$ are complex diagonal tensors of the form

$$\begin{aligned}[\vec{\mu}] &= \mu_0 \begin{pmatrix} \mu_x + \frac{\sigma_M^x}{j\omega} & 0 & 0 \\ 0 & \mu_y + \frac{\sigma_M^y}{j\omega} & 0 \\ 0 & 0 & \mu_z + \frac{\sigma_M^z}{j\omega} \end{pmatrix} \\ [\vec{\epsilon}] &= \epsilon_0 \begin{pmatrix} \epsilon_x + \frac{\sigma_E^x}{j\omega} & 0 & 0 \\ 0 & \epsilon_y + \frac{\sigma_E^y}{j\omega} & 0 \\ 0 & 0 & \epsilon_z + \frac{\sigma_E^z}{j\omega} \end{pmatrix}.\end{aligned}\quad (2)$$

The following condition is required to match the intrinsic impedance of the medium to free space

$$\frac{[\vec{\epsilon}]}{\epsilon_0} = \frac{[\vec{\mu}]}{\mu_0}.\quad (3)$$

Thus, $[\vec{\epsilon}]$ and $[\vec{\mu}]$ can be written as

$$\begin{aligned}[\vec{\mu}] &= \mu_0 [\Lambda] = \mu_0 \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \\ [\vec{\epsilon}] &= \epsilon_0 [\Lambda] = \epsilon_0 \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.\end{aligned}\quad (4)$$

Consequently, Maxwell's equations in the medium reduce to

$$\begin{aligned}\vec{\nabla} \cdot [\Lambda] \vec{E} &= 0 \\ \vec{\nabla} \cdot [\Lambda] \vec{H} &= 0 \\ \vec{\nabla} \times \vec{E} &= -j\omega\mu_0 [\Lambda] \vec{H} \\ \vec{\nabla} \times \vec{H} &= j\omega\epsilon_0 [\Lambda] \vec{E}.\end{aligned}\quad (5)$$

It can be shown that plane waves are eigensolutions of Maxwell's equations in the diagonally anisotropic medium. General solutions for $\vec{E}$ and $\vec{H}$ can be constructed from plane waves of the form

$$\begin{aligned}\vec{E}(\vec{r}; t) &= \vec{\mathcal{E}} e^{-j(\vec{k} \cdot \vec{r} - \omega t)} \\ \vec{H}(\vec{r}; t) &= \vec{\mathcal{H}} e^{-j(\vec{k} \cdot \vec{r} - \omega t)}\end{aligned}\quad (6)$$

where $\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$, and $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ are constant vectors. Substituting (6) into (5) results in

$$\begin{aligned}\vec{k} \cdot [\Lambda] \vec{\mathcal{E}} &= \vec{k} \cdot [\Lambda] \vec{\mathcal{H}} = 0 \\ \vec{k} \times \vec{\mathcal{E}} &= \omega\mu_0 [\Lambda] \vec{\mathcal{H}} \\ \vec{k} \times \vec{\mathcal{H}} &= -\omega\epsilon_0 [\Lambda] \vec{\mathcal{E}}.\end{aligned}\quad (7)$$

The dispersion relation which determines the form of the propagation vector $\vec{k}$ can be determined easily using the following variable transformations

$$\begin{aligned}\vec{\mathcal{E}}' &= [\Lambda]^{\frac{1}{2}} \vec{\mathcal{E}} \\ \vec{\mathcal{H}}' &= [\Lambda]^{\frac{1}{2}} \vec{\mathcal{H}} \\ \vec{k}' &= \frac{1}{\sqrt{abc}} [\Lambda]^{\frac{1}{2}} \vec{k}.\end{aligned}\quad (8)$$

Applying the transformations to (7) results in

$$\begin{aligned}\vec{k}' \cdot \vec{\mathcal{E}}' &= \vec{k}' \cdot \vec{\mathcal{H}}' = 0 \\ \vec{k}' \times \vec{\mathcal{E}}' &= \omega\mu_0 \vec{\mathcal{H}}' \\ \vec{k}' \times \vec{\mathcal{H}}' &= -\omega\epsilon_0 \vec{\mathcal{E}}'.\end{aligned}\quad (9)$$

Since $\vec{k}'$ is perpendicular to both $\vec{\mathcal{E}}'$ and $\vec{\mathcal{H}}'$, the dispersion relation can be obtained from (7) and is given by

$$\vec{k}' \cdot \vec{k}' = k_0^2 = \omega^2 \mu_0 \epsilon_0.\quad (10)$$

Finally, by using the inverse of the variable transformation in (8), the dispersion relation becomes

$$\frac{k_x^2}{bc} + \frac{k_y^2}{ac} + \frac{k_z^2}{ab} = k_0^2.\quad (11)$$

The dispersion relation (11) is the equation of an ellipsoid whose solutions are of the form

$$\begin{aligned}k_x &= k_0 \sqrt{bc} \sin \theta \cos \phi \\ k_y &= k_0 \sqrt{ac} \sin \theta \sin \phi \\ k_z &= k_0 \sqrt{ab} \cos \theta.\end{aligned}\quad (12)$$

This result suggests that the individual components of the propagation vector can be manipulated through the choice of a , b , and c .

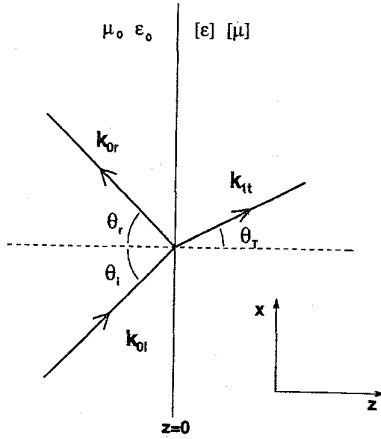


Fig. 2. Oblique incidence on a z -plane interface. In general, θ_T is complex.

III. REFLECTION COEFFICIENT

To determine the reflection coefficient of the interface in Fig. 1, a plane wave incidence problem must be solved. The propagation vectors are restricted to the xz -plane, so the dispersion relation (11) reduces to

$$\begin{aligned} k_x &= k_0 \sqrt{bc} \sin \theta \\ k_y &= 0 \\ k_z &= k_0 \sqrt{ab} \cos \theta. \end{aligned} \quad (13)$$

The geometry of the plane wave incidence problem is shown in Fig. 2. Any arbitrarily polarized plane wave can be decomposed into a linear combination of TE_y ($\vec{E}$ has only a y -component) and TM_y ($\vec{H}$ has only a y -component) modes. For the TE case, the incident, reflected, and transmitted waves are given by

$$\begin{aligned} \vec{E}_i(\vec{r}) &= \mathcal{E} \hat{y} e^{-jk_0(\sin \theta_i x + \cos \theta_i z)} \\ \vec{E}_r(\vec{r}) &= R^{TE} \mathcal{E} \hat{y} e^{-jk_0(\sin \theta_r x - \cos \theta_r z)} \\ \vec{E}_t(\vec{r}) &= T^{TE} \mathcal{E} \hat{y} e^{-jk_0(\sqrt{bc} \sin \theta_t x + \sqrt{ab} \cos \theta_t z)} \\ \vec{H}_i(\vec{r}) &= \sqrt{\frac{\epsilon_0}{\mu_0}} (-\cos \theta_i \hat{x} + \sin \theta_i \hat{z}) \\ &\quad \times e^{-jk_0(\sin \theta_i x + \cos \theta_i z)} \\ \vec{H}_r(\vec{r}) &= R^{TE} \sqrt{\frac{\epsilon_0}{\mu_0}} (\cos \theta_r \hat{x} + \sin \theta_r \hat{z}) \\ &\quad \times e^{-jk_0(\sin \theta_r x - \cos \theta_r z)} \\ \vec{H}_t(\vec{r}) &= T^{TE} \sqrt{\frac{\epsilon_0}{\mu_0}} \left(-\sqrt{\frac{b}{a}} \cos \theta_t \hat{x} + \sqrt{\frac{b}{c}} \sin \theta_t \hat{z} \right) \\ &\quad \times e^{-jk_0(\sqrt{bc} \sin \theta_t x + \sqrt{ab} \cos \theta_t z)} \end{aligned} \quad (14)$$

where R^{TE} and T^{TE} are the reflection and transmission coefficients for TE_y polarization.

Enforcing the tangential continuity of $\vec{E}$ and $\vec{H}$ on the interface involves matching the magnitude and phase of the E_y - and H_x -components. Magnitude matching results in

$$\begin{aligned} 1 + R^{TE} &= T^{TE} \\ \cos \theta_i - R^{TE} \cos \theta_i &= T^{TE} \sqrt{\frac{b}{a}} \cos \theta_t. \end{aligned} \quad (15)$$

The phase matching condition is a generalization of Snell's law and is given by

$$\begin{aligned} \sin \theta_i &= \sin \theta_r \\ \sqrt{bc} \sin \theta_t &= \sin \theta_i. \end{aligned} \quad (16)$$

Solving for R^{TE} in (15) and (16) gives

$$R^{TE} = \frac{\cos \theta_i - \sqrt{\frac{b}{a}} \cos \theta_t}{\cos \theta_i + \sqrt{\frac{b}{a}} \cos \theta_t}. \quad (17)$$

For the TM case, the same procedure can be followed to determine R^{TM} , resulting in

$$R^{TM} = \frac{\sqrt{\frac{b}{a}} \cos \theta_t - \cos \theta_i}{\cos \theta_i + \sqrt{\frac{b}{a}} \cos \theta_t}. \quad (18)$$

Imposing the requirement $\sqrt{bc} = 1$ simplifies the phase matching condition (16). It follows that $\theta_t = \theta_i$ and the reflection coefficients are not a function of incident angle. By also requiring $a = b$, the interface will be perfectly reflectionless for any frequency, angle of incidence, and polarization.

IV. MATERIAL PROPERTIES OF THE PML

The values of a , b , and c are not independent. They are related by

$$a = b = \frac{1}{c}. \quad (19)$$

Thus, the PML layer can then be characterized by one complex number $a = \alpha - j\beta$. When $\alpha, \beta > 0$, the transmitted wave will be damped in the anisotropic medium

$$\vec{E}_t(x, z) = \vec{E}_0 e^{-k_0 \beta \cos \theta_t z} e^{-jk_0(\sin \theta_t x + \alpha \cos \theta_t z)} e^{j\omega t}. \quad (20)$$

From (20), it is easy to see that α determines the wave length in the anisotropic absorber and β determines the rate of decay of the transmitted wave.

Now, it is interesting to interpret the material properties required for a perfectly matched interface. Using (4), the material tensor $[\Lambda]$ can be written as

$$[\Lambda] = \begin{pmatrix} \alpha - j\beta & 0 & 0 \\ 0 & \alpha - j\beta & 0 \\ 0 & 0 & \frac{\alpha + j\beta}{(\alpha^2 + \beta^2)} \end{pmatrix}. \quad (21)$$

The material properties (ϵ , μ , σ_E , σ_M) are now given by

$$\begin{aligned} [\epsilon] &= \epsilon_0 [P] \\ [\mu] &= \mu_0 [P] \\ [\sigma_E] &= \omega \epsilon_0 [S] \\ [\sigma_M] &= \omega \mu_0 [S] \end{aligned} \quad (22)$$

where

$$[P] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \frac{\alpha}{(\alpha^2 + \beta^2)} \end{pmatrix} \quad (23)$$

$$[S] = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\frac{\beta}{(\alpha^2 + \beta^2)} \end{pmatrix}. \quad (24)$$

Note that the z -component of the electric and magnetic conductivities is negative, implying the existence of dependent sources within the material.

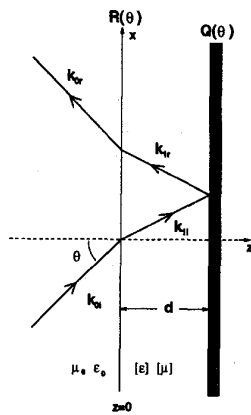


Fig. 3. Reflection from a finite slab of a perfectly matched anisotropic medium.

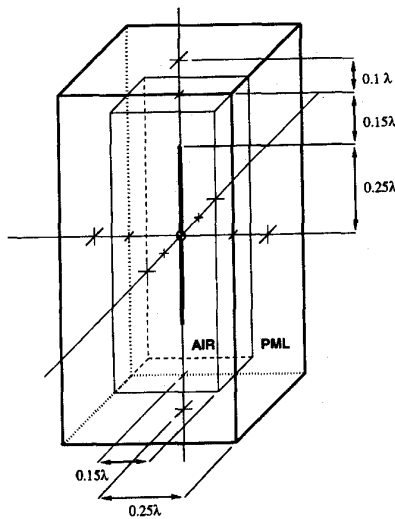


Fig. 4. Geometry of the half-wave dipole radiation problem.

V. FEM IMPLEMENTATION

In any practical implementation of the anisotropic PML, the absorbing layer must be truncated as shown in Fig. 3. The backing used to truncate the PML will have a nonzero, angle-dependent reflection coefficient $Q(\theta)$. The overall reflection coefficient of the anisotropic PML and its backing is given by

$$|R(\theta)| = |Q(\theta)|e^{-2\beta k_0 \cos \theta d}. \quad (25)$$

The anisotropic PML mesh truncation scheme was implemented in a FEM code using tetrahedral edge elements (lowest-order TVFEM). A perfectly conducting backing was used to truncate a homogeneous layer of the anisotropic PML.

VI. RESULTS

The far-field radiation pattern of a simple half-wave length dipole antenna was computed using a layer of anisotropic PML for mesh truncation. For this problem, the PML layer was placed 0.15λ away from the antenna element. The layer was 0.1λ thick. The geometry of the problem is shown in Fig. 4.

The FEM solution was obtained using a mesh with 4998 tetrahedra, where the mesh density was approximately 10 elements per wave

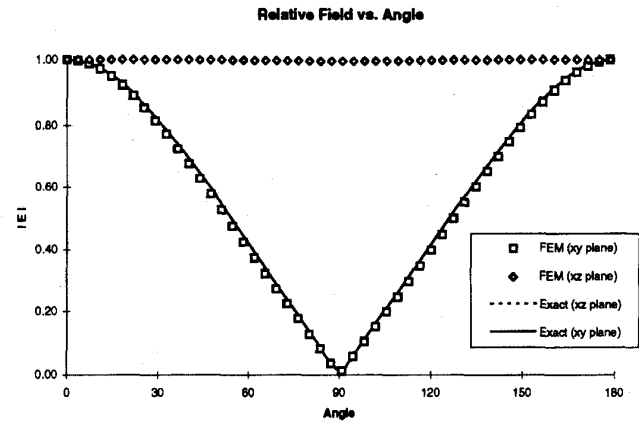


Fig. 5. Half-wave dipole far-field patterns obtained using anisotropic PML layer with $\beta = 3.0$.

length. The far-field pattern in two observation planes is plotted in Fig. 5. The results obtained using the anisotropic PML for mesh truncation are in good agreement with the exact solutions, even when using a relatively thin absorbing layer. In this case ($d = 0.1\lambda$), the absorber was approximately two elements thick.

VII. CONCLUSION

A new formulation of the PML mesh truncation scheme was developed. The formulation is based on using diagonally anisotropic material to construct a lossy absorbing layer. Although this approach is similar to the PML scheme investigated by others, it is not equivalent. Unlike Berenger's PML, the anisotropic PML does not require any modification of Maxwell's equations. The primary benefit of this approach is convenient implementation, especially in frequency-domain finite element codes.

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