

Lecture X

2D/3D EM modeling

In this lecture, we continue to study 2D and 3D EM modeling. The simplest method is the FDTD method on rectangular grids.

Part I. Basic (hard and soft) sources

1. What is a "hard source"?

The TM to z solution discussed in Lecture #9 can be treated as an exactly 2D field that exists between two parallel metal plates - see Fig. 1. The spacing between the plates is not critical; it may have any arbitrary value Δz . The vertical component of the electric field is not affected by those plates and is constant with respect to the vertical coordinate.

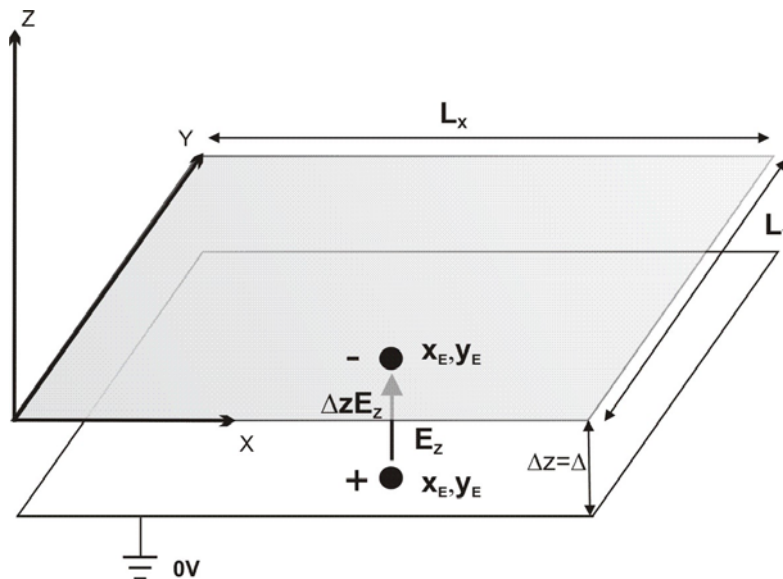


Fig. 1. An interpretation of a TM to z solution in the form of a 2D solution between two parallel (and maybe finite) metal plates.

The hard source or a "replaced" source 0 is equivalent to defining the vertical electric field at certain point, x_e, y_e , in the form

$$E_z(t, x_e, y_e) = \sin \omega t, \quad 0 < t < T_{\text{Off}} \quad (1a)$$

The replaced source was more commonly used in the past, for example in exciting coaxial and waveguide structures, but knowing which kind of source to use is important because their effects on the system are quite different. For example, replaced sources may cause reflections of waves propagating back to the source location. It can be shown that, in terms of a transmission-line representation - cf. Fig. 1 - the replaced source is analogous to placing a voltage source V , across a lossless transmission line.

2. How to turn the source on/off?

Because FDTD calculations give transient solutions, the turn-on characteristic of the source strongly affects the FDTD results and sometimes produces DC offsets and/or transients that delay the reaching of the steady-state response. In Reference cited below:

[D. N. Buechler, D. H. Roper, C. H. Durney, and D. A. Christensen, "Modeling sources in the FDTD formulation and their use in quantifying source and boundary condition errors," *IEEE Trans. Microwave Theory and Techniques*, vol. 43, no. 4, April 1995, pp. 810-814.]

different time functions were tested for the source waveform. It was found that using

$$E_z(t, x_e, y_e) = \left(\frac{1}{2} (1 - \cos(2\pi t / T_{\text{Off}})) \right) \sin \omega t, \quad 0 < t < T_{\text{Off}} \quad (1b)$$

with the raised cosine envelope produced the minimum amount of parasitic transients and no DC offset. Therefore, a raised cosine envelope waveform was suggested for all simulations.

To prove this conclusion, we perform a test in the script `fdtd_2d_02_HW.m` of this lecture. The hard source excitation in the script `fdtd_2d_02_HW.m` of this lecture is given either by Eq. (1a) or by Eq. (1b), with $T_{\text{Off}} = 2T$ where T is the period. Frequency is fixed at $f = 1 \text{ GHz}$. The geometry under study is a xy-domain $8\text{m} \times 6\text{m}$, with the spatial step of $\Delta x = \Delta y = 0.03\text{m}$. The hard source is located exactly in the middle. The effective "height" of the domain, Δz , is not explicitly included into the solution and therefore, plays no role.

Fig. 2 shows the two pulse forms; Fig. 3 gives the total energy found as a function of time. The shaped pulse in Fig. (2b) is slightly beneficial. The total energy is found by means of the code (accurate integration at the same time moment)

```
%-----
% Total field energy per 1 m in the vertical direction
% (averaged over every cell)
% EzN = zeros(Nx+1, Ny+1);
Energy_E = sum(sum(EzN(2:Nx, 2:Ny).^2)) ...
+ 0.5*sum(EzN(1, 2:Ny).^2) ...
+ 0.5*sum(EzN(Nx+1, 2:Ny).^2) ...
+ 0.5*sum(EzN(2:Nx, 1).^2) ...
+ 0.5*sum(EzN(2:Nx, Ny+1).^2) ...
+ 0.25*(EzN(1,1).^2 + EzN(1,Ny+1).^2 + EzN(Nx+1,1).^2 + EzN(Nx+1,Ny+1).^2);
% HxN = zeros(Nx+1, Ny+0);
% HyN = zeros(Nx+0, Ny+1);
Energy_H = sum(sum(HxN(2:Nx, 2:Ny-1).^2)) ...
```

```

+ 0.5*sum(HxN(1,      2:Ny-1).^2) ...
+ 0.5*sum(HxN(Nx+1,  2:Ny-1).^2) ...
+ 0.5*sum(HxN(2:Nx,  1).^2) ...
+ 0.5*sum(HxN(2:Nx,  Ny).^2) ...
+ 0.25*(HxN(1,1).^2 + HxN(1,Ny).^2 + HxN(Nx+1,1).^2 + HxN(Nx+1,Ny)).^2 ...
+ sum(sum(HyN(2:Nx-1,2:Ny)).^2) ...
+ 0.5*sum(HyN(1,      2:Ny).^2) ...
+ 0.5*sum(HyN(Nx,    2:Ny).^2) ...
+ 0.5*sum(HyN(2:Nx-1,1).^2) ...
+ 0.5*sum(HyN(2:Nx-1,Ny).^2) ...
+ 0.25*(HyN(1,1).^2 + HyN(1,Ny+1).^2 + HyN(Nx,1).^2 + HyN(Nx,Ny+1)).^2;
Energy(n) = 0.5*eps*Energy_E*d^2 + ...
            0.5*mu* Energy_H*d^2;

```

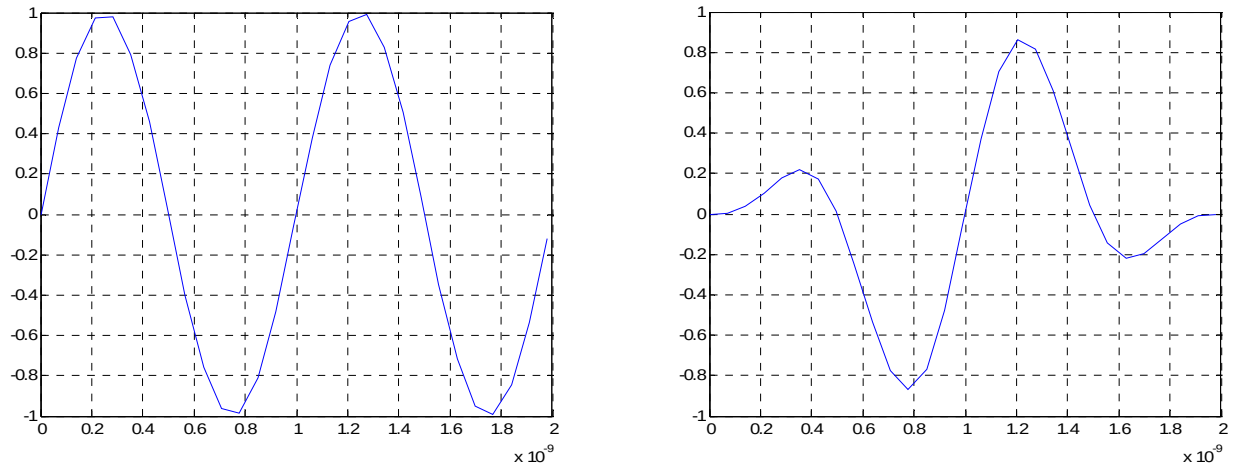


Fig. 2. Two pulse forms: left - two periods with no shaping. Right - raised cosine shaping.

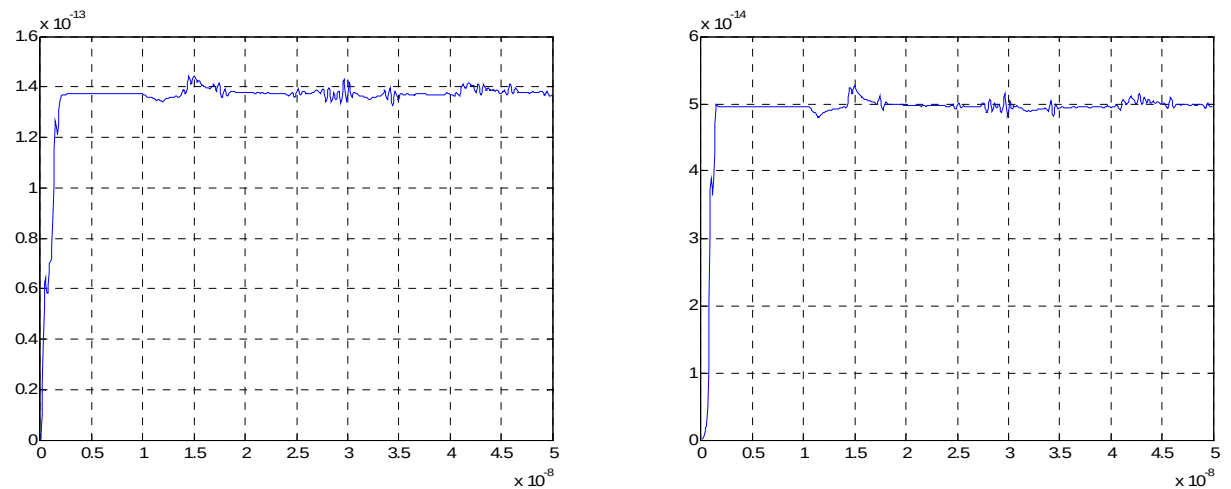


Fig. 3. Track of the total energy (J/m in the 2D case) in the cavity: left - two periods with no shaping. Right - raised cosine shaping. The energy stays constant to within approximately 3%.

The following observations can also be made:

1. Time-stepping (change in the time step) does not have a significant effect on the pulse distortion (and energy conservation).

2. Space resolution (change in the spatial step) does have an effect on the pulse distortion (and energy conservation), especially at 10 or less cells per wavelength

3. The pulse distortion that is observed for the hard source is generally the same as that for more realistic voltage/current soft sources considered below in the next lecture.

3. "Soft" current source

In this paragraph, we will follow the reference cited above. The Ampere's law (the equation for the E-field) can be rewritten in the form:

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon} \frac{\partial H_y}{\partial x} - \frac{1}{\epsilon} \frac{\partial H_x}{\partial y} - \frac{\sigma}{\epsilon} E_z \rightarrow \epsilon \frac{\partial E_z}{\partial t} = \nabla \times \vec{H} - J_z^c, \quad J_z^c = \sigma E_z \quad (2)$$

where J_z^c is the conduction current *density*, directed from the bottom metal plate to the top metal plate in Fig. 1. Similarly, one can introduce the impressed (or lumped) current density into Ampere's law in Maxwell's equations, as a source of EM radiation, i.e.

$$\epsilon \frac{\partial E_z}{\partial t} = \nabla \times \vec{H} - J_z^L, \quad J_z^L = \frac{I_z^L}{\Delta x \Delta y} \quad (3)$$

where I_z^L is the total lumped current. It is possible to rewrite Eq. (3) in terms of finite differences with the known lumped current as an added source or the *soft* source. The term *soft* thus means that we do not vary the fields directly; an extra source is rather added to Maxwell's equations.

The lumped impressed current density either exists in free space (as the Maxwell's displacement current in the capacitor), or it is supported by a physical conductor (a feed to a patch antenna) as shown in Fig. 4 that follows. In both cases, the initial source is still the voltage source. Therefore, we should rather consider the voltage sources in order to stay closer to reality.

In the following, when introducing the voltage source, we will denote the total lumped current by the generator current, i.e.

$$I_z^L = I_G \quad (4)$$

Note: Once the current source is introduced, we can find the E-field (z-component) at the same node and calculate the *impedance*. The impedance calculation has little sense in 2D, but is becoming vital in 3D. The idea is the same though.

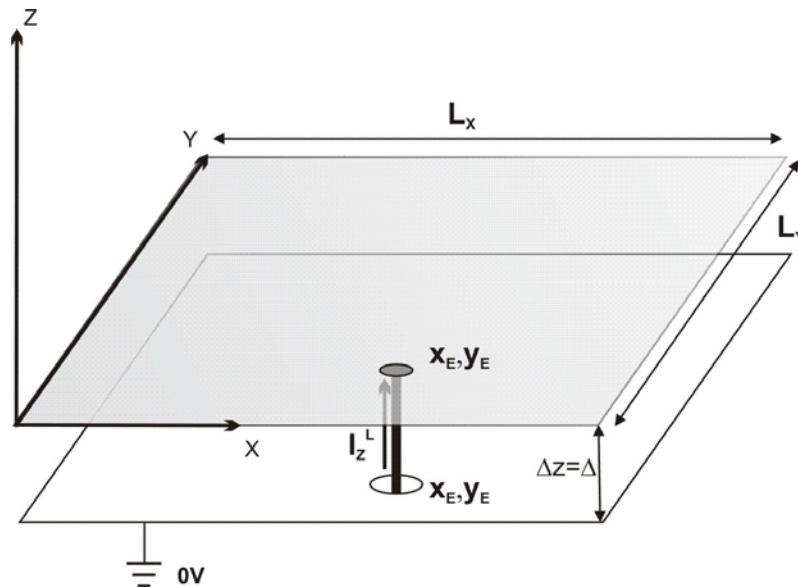


Fig. 4. One model of the current source - an unbalanced (coaxial) patch antenna feed.

Part II Boundary conditions

Absorbing boundary conditions (ABCs)

1 Problem statement and a short review

The metal box surrounding the structure is in fact not the only possibility for a FDTD solution. Usually, we are interested in the radiation problems when the radiated energy is dissipated, in same or other way, in the surrounding space. In this lecture, we discuss the FDTD Absorbing Boundary Conditions (ABCs) that are capable of absorbing such radiation energy. A review of different ABCs is given in Ref. [1]; see also Ref. [2]. In this lecture, we will implement the first- and second-order ABCs due to Mur [3].

- [1] A. Taflove, *Computational Electrodynamics, The Finite Difference Time Domain Approach*, Artech House, Norwood, MA, 1995.
- [2] A. Bondeson, T. Rylander, and P. Ingelström, *Computational Electromagnetics*, Springer, New York, 2005, Series: Texts in Applied Mathematics, Vol. 51., pp. 58-86.
- [3] G. Mur, "Absorbing boundary conditions for the finite-difference approximation of the time-domain electromagnetic field equations," *IEEE Trans. Electromagn. Compat.*, vol. EMC-23, no. 4, pp. 377-382, Nov. 1981.

2 First-order ABCs (Mur 1981)

Let's take a look at Fig. 5 that follows. First, if a source of excitation is located approximately in the center of the FDTD domain, and the size of this domain is large enough, the signal that hits the boundary is a combination of plane propagating waves. Not only that, at every planar boundary the signal is in fact an "almost" plane wave as shown in Fig. 5. In other words, the component of wave vector perpendicular to every boundary is dominant.

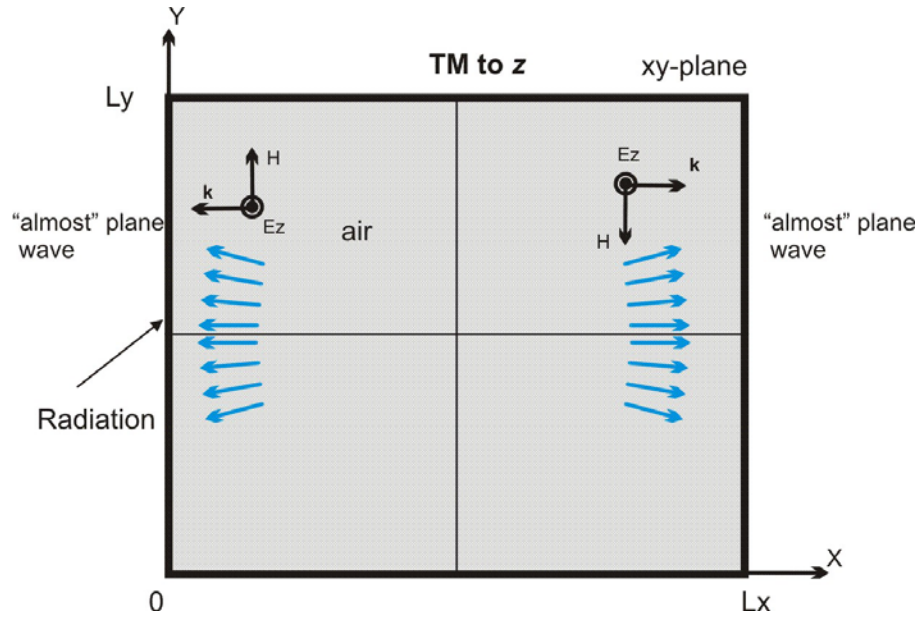


Fig. 5. An "almost" plane wave that is coming to the boundaries needs to be absorbed.

Such a field is conventionally described in terms of the so-called *parabolic* approximation, which initially was developed for well-collimated weakly-diffracted optical beams - almost plane waves. Let us start with the wave equation for an arbitrary field quantity, W ,

$$\frac{\partial^2 W}{\partial t^2} - c_0^2 \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right) = 0 \quad (5)$$

One can obtain another form of this equation, to underscore the dominant propagation along the x-axis

$$\left(\frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x} \right) W - c_0^2 \left(\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right) = 0, \quad (6)$$

either in the positive or in the negative direction. We are interested in the boundary at $x=0$, i.e. in the negative direction of propagation. When the direction of propagation is *exactly* the negative x -axis and the wave is *exactly* plane, Eq. (6) is reduced to

$$\left(\frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial x} \right) W = 0 \quad (7)$$

The first-order Mur's ABCs are given by Eq. (7) applied at all boundaries. In particular,

$$\frac{\partial E_z}{\partial t} - c_0 \frac{\partial E_z}{\partial x} = 0 \quad (8a)$$

$$\frac{\partial E_z}{\partial t} + c_0 \frac{\partial E_z}{\partial x} = 0 \quad (8b)$$

for the left and right boundary, respectively. The results for the lower and upper boundaries are obtained by permutation ($x \rightarrow y$). Despite the very simple nature, even those equations will do a decent job when implemented correctly.

Note: The first-order Mur's ABCs are given for the E-field only. The H-field is not yet involved.

3 Implementation of first-order ABCs

Let us start with the first-order Mur's ABCs Eqs. (8). The central point is how to implement them properly at the boundaries. According to Ref. [3] above, we will use the **centered differences** in both the space and the time increments (this is the key!), so that our result will have a local truncation error of the second order in all increments. One has in terms of notations from lecture #9:

$$\frac{\partial E_z}{\partial t} - c_0 \frac{\partial E_z}{\partial x} = 0 \quad (9a)$$

$$\frac{1}{2} \left(\frac{E_{z1,m}^{n+1} - E_{z1,m}^n}{\Delta t} + \frac{E_{z2,m}^{n+1} - E_{z2,m}^n}{\Delta t} \right) - c_0 \frac{1}{2} \left(\frac{E_{z2,m}^{n+1} - E_{z1,m}^{n+1}}{\Delta x} + \frac{E_{z2,m}^n - E_{z1,m}^n}{\Delta x} \right) = 0$$

$$E_{z1,m}^{n+1} = E_{z2,m}^n + \frac{c_0 \Delta t - \Delta x}{c_0 \Delta t + \Delta x} (E_{z2,m}^{n+1} - E_{z1,m}^n)$$

for the left boundary and

$$\frac{\partial E_z}{\partial t} + c_0 \frac{\partial E_z}{\partial x} = 0 \quad (9b)$$

$$\frac{1}{2} \left(\frac{E_{z N_x+1,m}^{n+1} - E_{z N_x+1,m}^n}{\Delta t} + \frac{E_{z N_x,m}^{n+1} - E_{z N_x,m}^n}{\Delta t} \right) + c_0 \frac{1}{2} \left(\frac{E_{z N_x+1,m}^{n+1} - E_{z N_x,m}^{n+1}}{\Delta x} + \frac{E_{z N_x+1,m}^n - E_{z N_x,m}^n}{\Delta x} \right) = 0$$

$$E_{z N_x+1,m}^{n+1} = E_{z N_x,m}^n + \frac{c_0 \Delta t - \Delta x}{c_0 \Delta t + \Delta x} (E_{z N_x,m}^{n+1} - E_{z N_x+1,m}^n)$$

for the right boundary. The corresponding MATLAB code is implemented into the main loop in the script `fdtd_2d_03_M1.m`. It reads

```
m1 = (c0*dt - d)/(c0*dt + d);
% Radiation BCs (Mur 1981, first order)
EzN(1, :) = EzP(2, :) + m1*(EzN(2, :) - EzP(1, :)); % x = 0;
EzN(Nx+1, :) = EzP(Nx, :) + m1*(EzN(Nx, :) - EzP(Nx+1, :)); % x = Lx;
EzN(:, 1) = EzP(:, 2) + m1*(EzN(:, 2) - EzP(:, 1)); % y = 0;
EzN(:, Ny+1) = EzP(:, Ny) + m1*(EzN(:, Ny) - EzP(:, Ny+1)); % y = Ly;
```

for all four boundaries. Note that in this code we assume the same spatial step, Δ , in either direction.

For a 6×6m domain with the raised-cosine shaped two-period long initial pulse at 1 GHz the script `fdtd_2d_03_M1.m` to this lecture gives the total energy behavior (in dB) shown in Fig. 7a. The vertical line is when the pulse fully leaves the computational domain.

The field is tracked over a large period of time including the time moment, T_{off} , when the pulse completely leaves the box (along its diagonal). The energy remaining in the box by that time is the error due to non-perfect boundary conditions. The ratio of that energy and the maximum total energy with the entire pulse in the box, will give us the boundary condition performance or the *isolation*. This ratio should be expressed in dB. In the present case, the isolation is about 25 dB.

4 Second-order ABCs (Mur 1981). Method of parabolic equation

Those ABCs is a natural extension of the first-order ABCs. When this observation (of a plane wave given by Eq. (7)) is only approximately true, we could still replace one spatial derivative in the first term on the left-hand side of Eq. (6) by

$$\frac{\partial}{\partial t} \approx c_0 \frac{\partial}{\partial x} \Rightarrow \left(\frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x} \right) \approx 2 \frac{\partial}{\partial t} \quad (10)$$

Therefore, we have

$$\left(\frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial x} \right) \left(2 \frac{\partial}{\partial t} \right) W - c_0^2 \left(\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right) = 0 \quad (11)$$

or, which is the same,

$$\frac{\partial^2 W}{\partial t^2} - c_0 \frac{\partial^2 W}{\partial t \partial x} - \frac{c_0^2}{2} \left(\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right) = 0 \quad (12)$$

Eq. (12) is the celebrated *parabolic* approximation to wave equation, which is well known in optics. It says that the electromagnetic signal propagates predominantly along the negative x-axis; it is also a subject to diffraction in the transversal plane (in the yz-plane). The parabolic equation is easier to solve than the wave equation itself, and it is straightforward to formulate the boundary conditions in terms of it.

5 Application to TM to z fields

For the TM to z field from Lecture #9 (without magnetic loss),

$$\begin{aligned} \frac{\partial E_z}{\partial t} &= \frac{1}{\epsilon} \frac{\partial H_y}{\partial x} - \frac{1}{\epsilon} \frac{\partial H_x}{\partial y} - \frac{\sigma}{\epsilon} E_z \\ \frac{\partial H_x}{\partial t} &= -\frac{1}{\mu} \frac{\partial E_z}{\partial y} \\ \frac{\partial H_y}{\partial t} &= \frac{1}{\mu} \frac{\partial E_z}{\partial x} \end{aligned} \quad (13)$$

Now, Eq. (6) at the boundary is applied to the Ez-field only, i.e.

$$\frac{\partial^2 E_z}{\partial t^2} - c_0 \frac{\partial^2 E_z}{\partial t \partial x} - \frac{c_0^2}{2} \frac{\partial^2 E_z}{\partial y^2} = 0 \quad (14)$$

Substituting the field derivative with respect to y from Eqs. (13) and cancelling one time derivative, we arrive at

$$\frac{\partial E_z}{\partial t} - c_0 \frac{\partial E_z}{\partial x} + \mu \frac{c_0^2}{2} \frac{\partial H_x}{\partial y} = 0 \quad (15a)$$

Eq. (9a) describes the outgoing quasi plane wave at the left boundary in Fig. 5. Once programmed at this boundary, it will give us the corresponding radiation boundary condition. For the right boundary in Fig. 5,

$$\frac{\partial E_z}{\partial t} + c_0 \frac{\partial E_z}{\partial x} + \mu \frac{c_0^2}{2} \frac{\partial H_x}{\partial y} = 0 \quad (15b)$$

In other words, the propagation speed changes its sign. Eqs. (9) are known as Mur's second-order ABCs (Ref. [3] above). The first-order Mur's ABCs also exist, they are:

For completeness, we present here the second-order equations for the lower and upper boundaries, respectively

$$\frac{\partial E_z}{\partial t} - c_0 \frac{\partial E_z}{\partial y} - \mu \frac{c_0^2}{2} \frac{\partial H_y}{\partial x} = 0 \quad (16a)$$

$$\frac{\partial E_z}{\partial t} + c_0 \frac{\partial E_z}{\partial y} - \mu \frac{c_0^2}{2} \frac{\partial H_y}{\partial x} = 0 \quad (16b)$$

6 Implementation

Again, we assume the same spatial step, Δ , in either direction. According to Eqs. (15), (16), the second-order Mur's difference equations have the form

$$\begin{aligned} E_{z1,m}^{n+1} &= E_{z2,m}^n + m_1 (E_{z2,m}^{n+1} - E_{z1,m}^n) - m_2 (H_{x1,m+1/2}^{n+1/2} - H_{x1,m-1/2}^{n+1/2} + H_{x2,m+1/2}^{n+1/2} - H_{x2,m-1/2}^{n+1/2}) \\ E_{zN_x+1,m}^{n+1} &= E_{zN_x,m}^n + m_1 (E_{zN_x,m}^{n+1} - E_{zN_x+1,m}^n) - m_2 (H_{xN_x,m+1/2}^{n+1/2} - H_{xN_x,m-1/2}^{n+1/2} + H_{xN_x+1,m+1/2}^{n+1/2} - H_{xN_x+1,m-1/2}^{n+1/2}) \\ E_{zk,1}^{n+1} &= E_{zk,2}^n + m_1 (E_{zk,2}^{n+1} - E_{zk,1}^n) + m_2 (H_{yk+1/2,1}^{n+1/2} - H_{yk-1/2,1}^{n+1/2} + H_{yk+1/2,2}^{n+1/2} - H_{yk-1/2,2}^{n+1/2}) \\ E_{zk,N_y+1}^{n+1} &= E_{zk,N_y}^n + m_1 (E_{zk,N_y}^{n+1} - E_{zk,N_y+1}^n) + m_2 (H_{yk+1/2,N_y}^{n+1/2} - H_{yk-1/2,N_y}^{n+1/2} + H_{yk+1/2,N_y+1}^{n+1/2} - H_{yk-1/2,N_y+1}^{n+1/2}) \end{aligned} \quad (17)$$

where

$$m_1 = \frac{c_0 \Delta t - \Delta}{c_0 \Delta t + \Delta}, m_2 = \frac{\mu c_0^2 \Delta t / 2}{c_0 \Delta t + \Delta} \quad (18)$$

The stumbling stone are the corners were we still employ the first-order ABCs.

For a 6×6m domain with the two-period long raised-cosine shaped initial pulse at 1 GHz the script `fdtd_2d_03_M2.m` to this lecture gives the total energy behavior (in dB) shown in Fig. 7b. The vertical line is when the pulse fully leaves the computational domain. We can observe the improvement in isolation of about 12dB compared to Fig. 7a.

Note: The typical improvement values are however a bit less - about -7dB to -10dB.

7 “Superabsorption” ABCs (Mei et al 1992)

The Mei-Fang “superabsorption” method described in

K. K. Mei and J. Fang, “Superabsorption – a method to improve absorbing boundary conditions,” *IEEE Trans. Antennas and Propagation*, vol. 40, no. 9, Sep. 1992, pp. 1001-1010.

is not an ABC by itself, but rather a numerical procedure for the improvement of the local ABC's applied to the FDTD technique. It embodies an error-canceling formulation according to which the same ABC is applied to both E and H field components on and near the outer boundaries, depending on the polarization examined.

The calculation of the 2-D TM (TE) magnetic (electric) components, from their respective boundary ABC-derived electric (magnetic) ones, yields reflection errors which are strongly related to the errors in magnetic (electric) field components directly computed from the ABC. The opposite sign that these errors have in both of the above separate calculations is a point of crucial importance in the superabsorption procedure. Taking this fact into consideration and by properly combining the two different computations of the magnetic (electric) fields near the boundary, it is possible to cancel the reflection errors mutually while maintaining the correct values of the fields on the boundary.

Fig. 6 shows schematically the implementation of the method for the right boundary ($x=L$) of the computational domain.

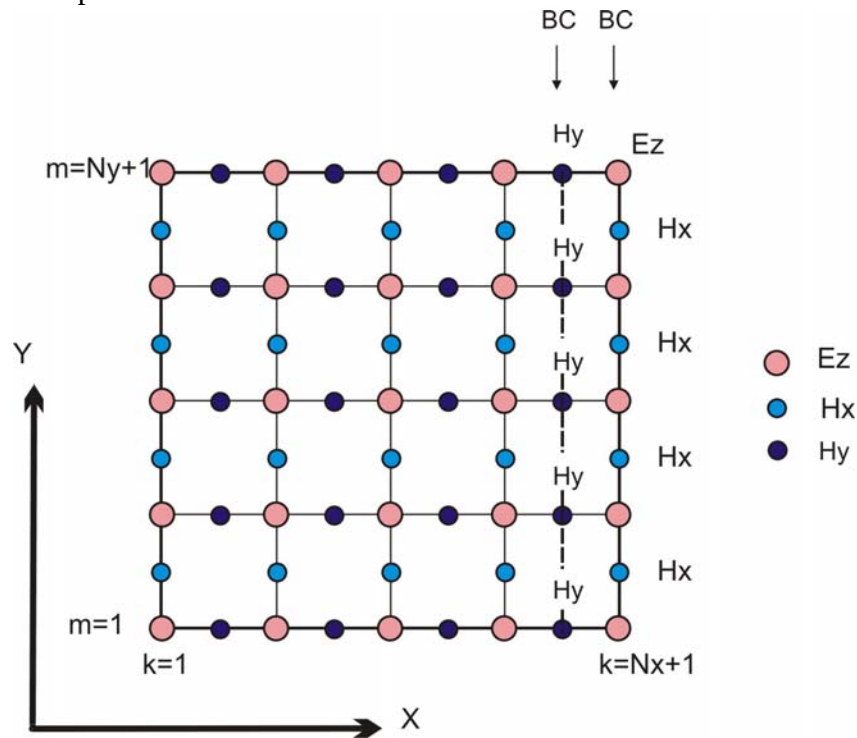


Fig. 6. The superabsorption BCS on the right boundary.

For this boundary, we apply the first-order Mur's ABC given by Eq. (8b) not only to the E-field but also to the Hy-field close to that boundary, i.e.

$$H_{y N_x, m}^{n+1/2(2)} = H_{y N_x, m}^{n-1/2} + \frac{c_0 \Delta t - \Delta x}{c_0 \Delta t + \Delta x} \left(H_{y N_x-1, m}^{n+1/2} - H_{y N_x, m}^{n-1/2} \right) \quad (19a)$$

for the right boundary. Next, we compute the Hy-field by the regular finite-difference scheme to obtain

$$H_{y_{N_x,m}}^{n+1/2(1)} \quad (19b)$$

After that, we build a weighted average of those two values and obtain the final updated magnetic field value at the last point by

$$H_{y_{N_x,m}}^{n+1/2} = \frac{H_{y_{N_x,m}}^{n+1/2(1)} + \rho H_{y_{N_x,m}}^{n+1/2(2)}}{1 + \rho} \quad (19c)$$

Here,

$$\rho = \frac{c_0 \Delta t}{\Delta} \quad (20)$$

It can be shown that this procedure significantly decreases the error of a local ABC, in particular, the first-order Mur's ABC. It is also very simply to implement and does not require any extra variables. The corresponding MATLAB code is given by

```
%-----
% Super ABCs for H (Mei 1992, first->second order)
% H(2)
HyP(1, :) = HyP(2, :) + m1*(HyN(2, :) - HyP(1, :)); % x = 0;
HyP(Nx, :) = HyP(Nx-1, :) + m1*(HyN(Nx-1, :) - HyP(Nx, :)); % x = Lx;
HxP(:, 1) = HxP(:, 2) + m1*(HxN(:, 2) - HxP(:, 1)); % y = 0;
HxP(:, Ny) = HxP(:, Ny-1) + m1*(HxN(:, Ny-1) - HxP(:, Ny)); % y = Ly;

% H(1) and H
HyN(1, :) = (rho*HyP(1, :) + HyN(1, :))/(1+rho); % x = 0;
HyN(Nx, :) = (rho*HyP(Nx, :) + HyN(Nx, :))/(1+rho); % x = Lx;
HxN(:, 1) = (rho*HxP(:, 1) + HxN(:, 1))/(1+rho); % y = 0;
HxN(:, Ny) = (rho*HxP(:, Ny) + HxN(:, Ny))/(1+rho); % y = Ly;
%-----
```

For a 6×6m domain with the raised-cosine shaped two-period long initial pulse at 1 GHz the script `fdtd_2d_03_M3.m` to this lecture gives the total energy behavior (in dB) shown in Fig. 7c. The first-order ABCs and the superabsorption are employed. The vertical line is when the pulse fully leaves the computational domain. We can observe the improvement of about 10 dB compared to Fig. 7a.

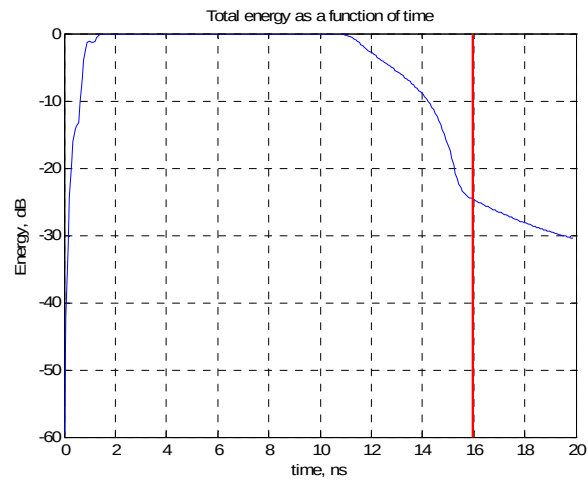
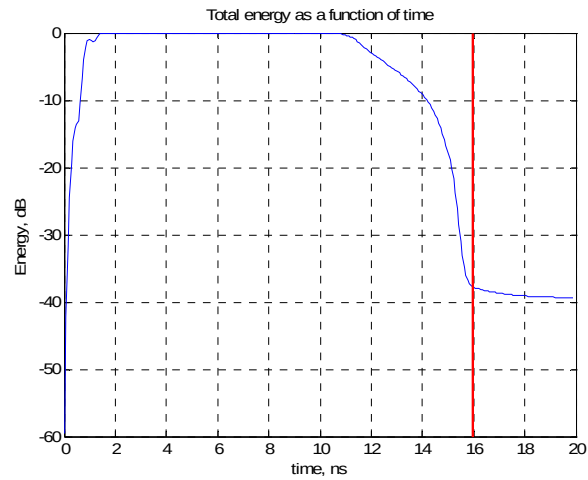
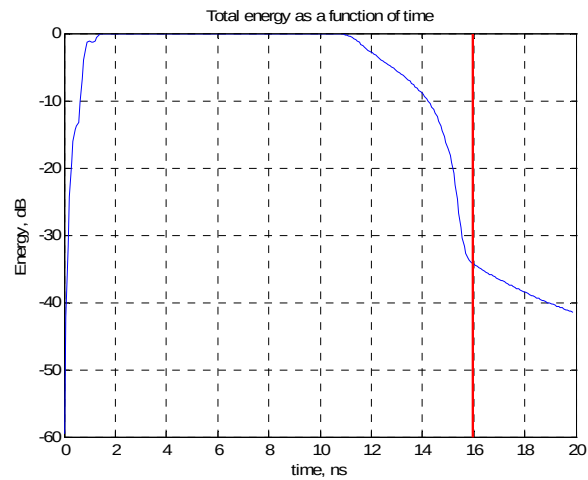
a) Mur's first-order ABCs**b) Mur's second-order ABCs****c) Mur's first-order ABCs + superabsorbing ABCs**

Fig. 7. Energy isolation (dB) for three local ABCs.

Part III Boundary conditions

"His excellence, the PML"

1 Problem statement and a short review

The ABCs discussed above in this lecture are not the only possibility for a FDTD radiation solution. Often, we are interested in the radiation problems with higher isolation than that provided by the ABCs. In this part, we discuss the FDTD Perfectly Matched Layer (PML) that is capable of giving about -80 dB or so noise floor due to artificial reflections.

Although well described in the textbooks [1],[2], the PML is perhaps best studied using the original papers, in particular Refs. [3]-[12]. The PML was first discovered by J.-P. Bérenger [3]-[8]. Then, it was extensively tested by Allen Taflove and co-workers - see Refs. [1],[9],[10]. Only one year after the initial series of fundamental papers [3],[9],[10], Sacks *et al* [11] published an independent model of the PML, the so-called anisotropic or unsplit PLM, which quickly became the **de-facto standard** not only for FDTD simulations but also in the finite-element method [7],[11],[12]. In particular, the Ansoft HFSS is using the anisotropic PML *similar* to that described in Ref. [11]. The idea of the PML is different from ABCs - the PML rather absorbs fields than let them go.

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- [7] J.-P. Bérenger, "Numerical reflection from FDTD-PML's: a comparison of the split PML with the unsplit and CFS PML's," *IEEE Trans. Antennas Propagat.*, vol. 50, pp. 258–265, 2002.
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[11] Z. S. Sacks, D. M. Kingsland, R. Lee, and J.-F. Lee, "A perfectly anisotropic absorber for use as an absorbing boundary condition," *IEEE Trans. Antennas Propagat*, vol. 43, pp. 1460-1463, Dec. 1995.

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[13] P. G. Pteropoulus, L. Zhao, and A. C. Cangellaris, "A reflectionless sponge layer absorbing boundary condition for the solution of Maxwell's equations with high-order staggered finite difference scheme," *J. Comp. Physics*, vol. 139, pp. 184-208, 1998.

2 What does the PML do?

For simplicity, we start with the 2D field again, namely, with the TE to z field shown in Fig. 8 that follows. The magnetic field in this case is always directed along the z -axis, but the propagation direction (wave vector) is always in the xy -plane.

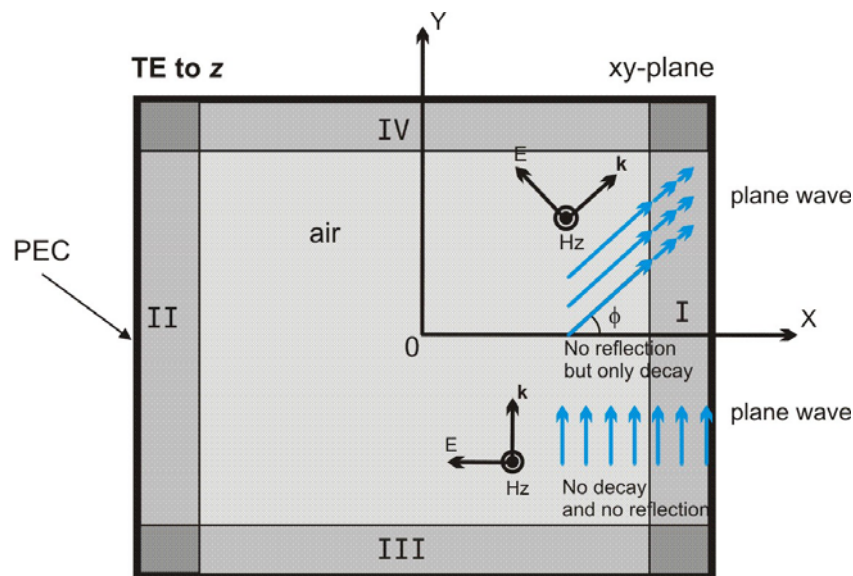


Fig. 8. The PML concept for the TE to z field.

The PML in Fig. 8 should satisfy two conditions:

1. Absorb a plane wave at any incidence angle, without reflection from the PML boundary at any incidence angle. Absorption and reflectionless for any incidence angle initiated the word "perfectly" in the PML abbreviation. The incidence must not be exactly oblique.
2. Do not absorb and do not perturb any plane wave at exactly oblique incidence -see Fig. 8.

The latter condition is conventionally satisfied if we terminate the PML into a PEC boundary, that is a perfect electric conductor shown in Fig. 8.

The next case is the TM to z case, considered in the previous lecture, and also shown in Fig. 9. The electric field in this case is always directed along the z -axis, but the propagation direction (wave vector) is again always in the xy -plane.

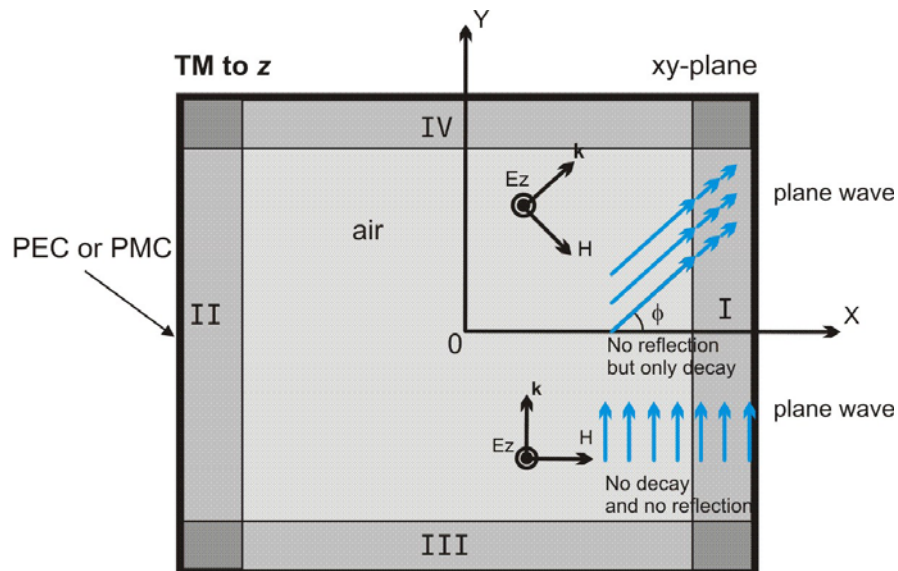


Fig. 9. The PML concept for the TM to z field.

If we require both conditions 1 and 2 to be satisfied, then it is seen from geometry of Fig. 9 that a PEC boundary of the PML will disturb the solution at oblique incidence. The PMC (perfect magnetic conductor - the tangential H -component is zero) boundary or the radiation boundary should be actually more adequate for this case. If the wave signal at oblique incidences is not present or is not important, the difference between the boundaries is not significant. Both boundaries can be equally simply implemented in the FDTD method.

3 Why does the PML have multiple regions?

In all distinct PML regions shown in Fig. 8 and Fig. 9 the loss parameters of an artificial lossy media will be different, irrespectively of the particular PML model used - see below in this lecture. In fact, the PML was first carefully formulated for the TE case in Fig. 8, by assigning those distinct parameters.

4 How does Bérenger's model work?

The PML boundary condition is a lossy material boundary layer that is perfectly matched to the physical solution space. In the original Bérenger's model [3],[4],[5] this is achieved through a "field splitting" of the electric and magnetic field intensities, leading to a modified set of Maxwell's equations. It has been shown that an arbitrarily polarized wave

incident on this PML medium is perfectly transmitted, has the same phase velocity and characteristic wave impedance as the incident wave while attenuating rapidly along the normal axis. Béranger's PML method has been successfully implemented within the FDTD algorithm [12].

There are limitations of the present form of Béranger's PML medium, however. Principally, the fields within the PML medium and the governing equations are *non-Maxwellian*. This in itself does not detract from the usefulness of the method, however, it impedes direct physical insight into the mechanisms that render it effective [12]. For example, three Eqs. (1) of Lecture #9 for the TM to z field are now split into four equations in the PML region [3]:

$$\begin{aligned}
 \frac{\partial E_z}{\partial t} &= \frac{1}{\epsilon} \frac{\partial H_y}{\partial x} - \frac{1}{\epsilon} \frac{\partial H_x}{\partial y} - \frac{\sigma}{\epsilon} E_z \\
 \frac{\partial H_x}{\partial t} &= -\frac{1}{\mu} \frac{\partial E_z}{\partial y} - \frac{\sigma^*}{\mu} H_x \\
 \frac{\partial H_y}{\partial t} &= \frac{1}{\mu} \frac{\partial E_z}{\partial x} - \frac{\sigma^*}{\mu} H_y
 \end{aligned}
 \Rightarrow
 \begin{aligned}
 \frac{\partial E_{zx}}{\partial t} &= \frac{1}{\epsilon} \frac{\partial H_y}{\partial x} - \frac{\sigma_x}{\epsilon} E_{zx} \\
 \frac{\partial E_{zy}}{\partial t} &= -\frac{1}{\epsilon} \frac{\partial H_x}{\partial y} - \frac{\sigma_y}{\epsilon} E_{zy} \\
 \frac{\partial H_x}{\partial t} &= -\frac{1}{\mu} \frac{\partial (E_{zx} + E_{zy})}{\partial y} - \frac{\sigma_y^*}{\mu} H_x \\
 \frac{\partial H_y}{\partial t} &= \frac{1}{\mu} \frac{\partial (E_{zx} + E_{zy})}{\partial x} - \frac{\sigma_x^*}{\mu} H_y
 \end{aligned}
 \quad (21)$$

by splitting the electric field into two independent subcomponents, $E_z \rightarrow E_{zx} + E_{zy}$ in the PML region. The Maxwell's equations themselves give no reflection at the boundary at normal incidence if [1],[2]

$$\frac{\sigma}{\epsilon} = \frac{\sigma^*}{\mu} \quad (22)$$

However, the reflection at other incidence angles will still be present. On the other hand, their counterpart from Eqs. (21), which matches to the original Maxwell's equations at the boundary where

$$E_z = E_{zx} + E_{zy} \quad (23)$$

gives no reflections and satisfies conditions 1 and 2 for the PML by a proper choice of artificial conductivities $\sigma_{x,y}, \sigma_{x,y}^*$. In particular, in both domains I and II in Fig. 1 and 2 [3], only two loss coefficients are present

$$\frac{\sigma_x}{\epsilon} = \frac{\sigma_x^*}{\mu} \neq 0, \quad \sigma_y = \sigma_y^* = 0 \quad (24a)$$

Similarly, in both domains III and IV in Figs. 8, 9

$$\frac{\sigma_y}{\varepsilon} = \frac{\sigma_y^*}{\mu} \neq 0, \quad \sigma_x = \sigma_x^* = 0 \quad (24b)$$

At the four corners of the PML, where there is overlap of two subdomains [3], all four losses are present, i.e.

$$\frac{\sigma_x}{\varepsilon} = \frac{\sigma_x^*}{\mu} \neq 0, \quad \frac{\sigma_y}{\varepsilon} = \frac{\sigma_y^*}{\mu} \neq 0 \quad (24c)$$

This is done to match the PML subdomains to each other. A similar scheme is used in the general 3D case [4].

Despite some initial complexity, the practical FDTD implementation of Eqs. (21) is straightforward and it is described in detail in Refs. [3],[4] cited above. For the 2D case, three fields are updated marching on in time in the internal region, and four fields in the PML region, respectively. The update at the PML interface specifically described in Ref. [3] connects all fields together.

5 PML implementation[3]

In this section, we recall several items that may be important for flawless PML implementation and operation.

5.1 Averaging [3]

According to Ref. [3], we compute all conductivities as the average values in the cell around the index location. This can be written in the form

$$\sigma_x(k) = \frac{1}{\Delta x} \int_{x(k)-\Delta x/2}^{x(k)+\Delta x/2} \sigma_x(x') dx', \quad (25a)$$

etc. An attempt to average the FDTD coefficients themselves (which is easier to implement) will result in PML failure.

5.2 Exponential update [3],[7]

To achieve an unconditional stability at any values of the conductivity, we use the so-called exponential update

$$\begin{aligned}
e_{z1} &= \frac{1-\tau/2}{1+\tau/2} \rightarrow \exp(-\tau), & e_{z2} &= \frac{\Delta t}{\varepsilon_{k,m}\Delta} \frac{1}{1+\tau/2} \rightarrow \frac{\Delta t}{\varepsilon_{k,m}\Delta} \frac{1-\exp(-\tau)}{\tau}, & \tau &= \frac{\Delta t \sigma}{\varepsilon_{k,m}} \\
h_1 &= \frac{1-\tau^*/2}{1+\tau^*/2} \rightarrow \exp(-\tau^*), & h_2 &= \frac{\Delta t}{\mu_{k,m+1/2}\Delta} \frac{1}{1+\tau^*/2} \rightarrow \frac{\Delta t}{\mu_{k,m+1/2}\Delta} \frac{1-\exp(-\tau^*)}{\tau^*}, & \tau^* &= \frac{\Delta t \sigma^*}{\mu_{k,m+1/2}}
\end{aligned}
\tag{25b}$$

that is based on a partial solution of the lossy Maxwell's equations. Otherwise, at high values of the conductivity, the above coefficients may become very large and then negative. The exponential update should be also used for *any highly-absorbing medium*, not necessarily the PML.

5.3 Conductivity profile [3],[7]

We use the "standard" quadratic Bérenger's conductivity profile (a cubic profile is more common now)

$$\sigma = \sigma_{\max} \left(\frac{r}{R_{\text{PML}}} \right)^n, \quad n = 2, \quad \frac{\sigma}{\varepsilon} = \frac{\sigma^*}{\mu}
\tag{25c}$$

where R_{PML} is PML thickness. If a constant conductivity over the layer is employed, the FDTD reflection coefficient cannot be smaller than 3% [3]. This value decreases to 0.1% for a linear profile, and to 0.01% for a quadratic profile.

5.4 Reflection coefficient

The theoretical reflection coefficient is given by

$$R = \exp \left(- \frac{2}{n+1} \frac{\sigma_{\max} R_{\text{PML}}}{\varepsilon c_0} \right)
\tag{25d}$$

where R_{PML} is PML thickness. Interestingly enough, the reflection coefficient only depends on PML thickness, not the number of cells within the PML. This means that the PML should theoretically work for any number of cells; a practical value is, however, about 10-15. The optimum values for the reflection coefficient are 10^{-5} to 10^{-6} [3].

5.5 Separate or not the PML from the main scheme?

The implementation is easier if the PML is not separated from the main field and the entire medium is the "PML medium" with certain local values of conductivities, being zero in the main domain. However, that method results in a potentially slower algorithm, due to necessity of extra multiplications. Therefore, the separated PML that is shown in Fig. 10 would be preferred.

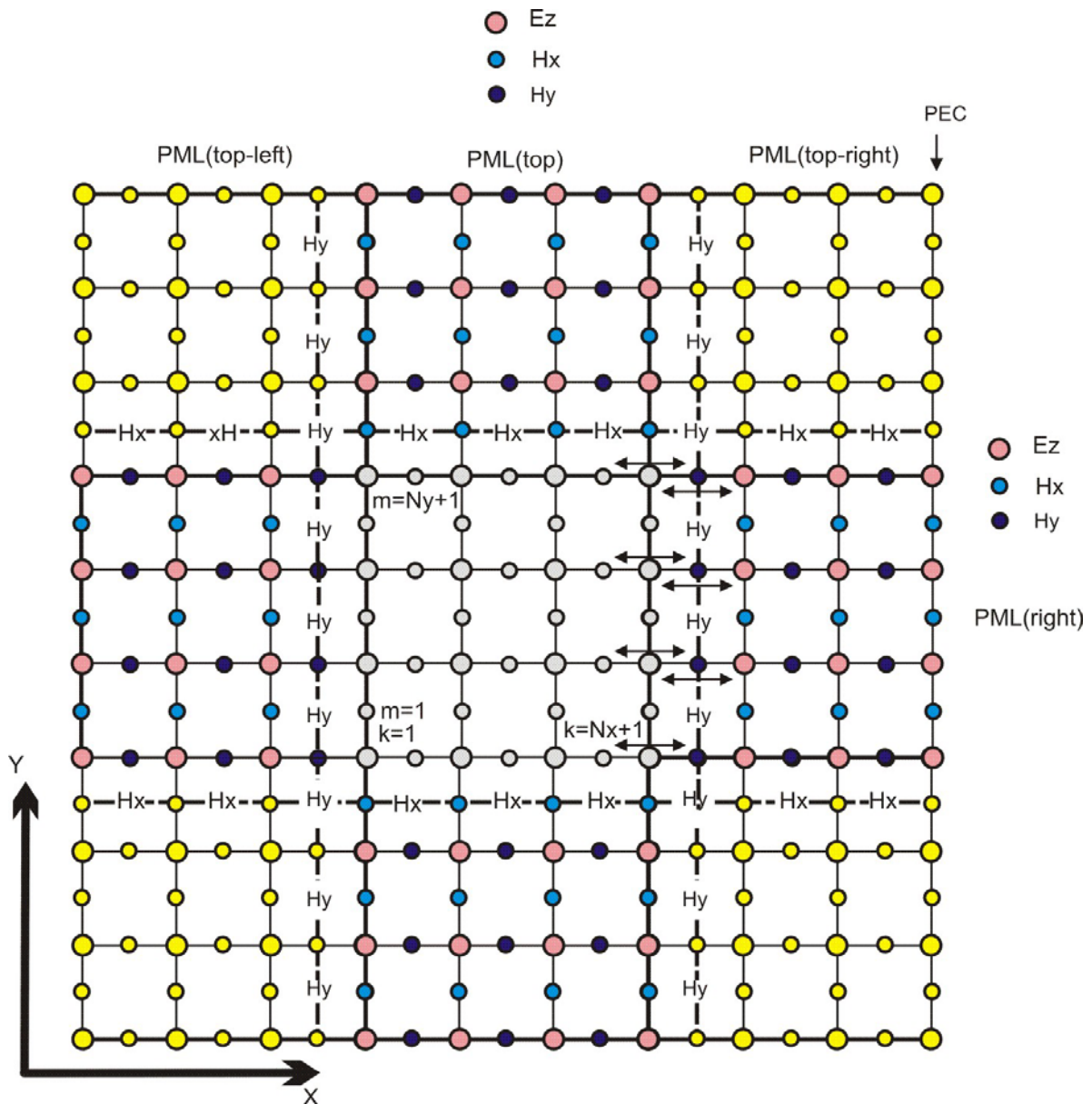


Fig. 10. A sketch of one PML grid with sides and corners. The PML is separated from the main domain. This figure has been used to build the 2D PML for ECE539.

When programming the separated PML, the fields should be matched at all the boundaries, using E or H values from different domains if necessary.

5.6 Every corner is important!

Even a simple mistake in the PML (one corner or so) will result in a drastic degradation of the performance.

6 PML performance

For the same 6×6 m domain with the raised-cosine shaped two-period long initial pulse at 1 GHz the script `fdtd_2d_03_M4.m` to this lecture gives the total energy behavior (in dB) shown in Fig. 11. This script employs the 2D Béranger's PML with 15 nodes (in width).

We can observe some improvement compared to different ABCs in Fig. 7. However, this improvement is not really drastic, only about 10 dB compared to the best case (the second-order ABCs). What's the matter? The answer is in the remaining isolation noise due to

- A. A non-perfect "hard" excitation source (the major reason);
- B. A non-perfect numerical scheme;

with about -50 dB noise floor or so. When the noise floor is carefully controlled the isolation reaches 80 dB or so.

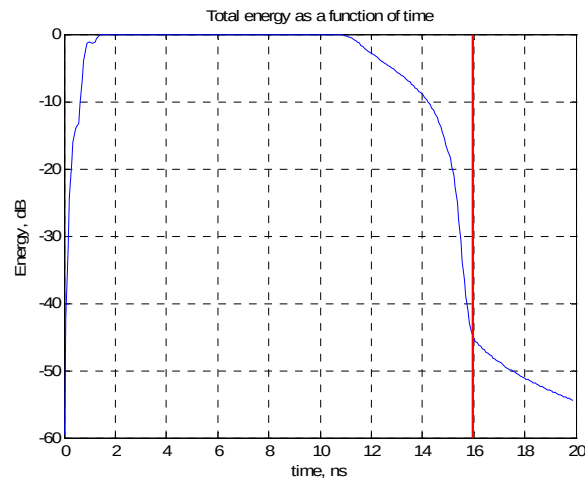


Fig. 11. Energy isolation (dB) for the PML with 15 nodes.

Note:

Discuss the Ansoft radiating conditions
Discuss the Ansoft PML (start with the history)

HW#10 exercise

1. 3D FDTD - ABCs

Using the basic script `fdtd_3d_01.m` to this lecture (a plain 3D FDTD algorithm) please implement the first-order ABCs in 3D. To do so:

- A. Carefully write the analog of Eqs. (9) (the appropriate PDE and the corresponding difference equation) for each boundary. Remember that there are 6 boundaries total.
- B. Accurately program the resulting difference equations.
- C. Plot the energy behavior (within the computational domain) to scale over the time interval from 0 to 8 ns (insert the grid into the plot) with and without the ABCs.

Appendix A

1 How does the anisotropic model work?

The anisotropic PML model introduced by Sacks, Kingsland, R. Lee, and J.-F. Lee [11] only one year after Bérenger's model performs quite differently, but leads to the same results as the Bérenger's PML [7],[12] (see the references cited above). It is in a form that avoids the field splitting.

The advantage of this formulation over Berenger's PML method are: 1) the formulation is based on Maxwell's equations, rather than a modified set of equations, 2) the application to the FDTD method is more computationally efficient, and 3) it can be extended to non-orthogonal and unstructured grid techniques [12]. We will present the model for the anisotropic PML layer described in Refs. [11],[7], [12] (cited above).

In order to start with, we need to consider Maxwell's equations in an arbitrary (lossy, anisotropic, dispersive) but still *linear* medium. Maxwell's equations in the form in which they can be found in most textbooks on electromagnetics read

$$\begin{aligned}\frac{\partial \vec{D}}{\partial t} &= \nabla \times \vec{H} - \vec{J} \\ \frac{\partial \vec{B}}{\partial t} &= -\nabla \times \vec{E} \\ \nabla \cdot \vec{D} &= \rho \\ \nabla \cdot \vec{B} &= 0\end{aligned}\tag{A1}$$

where $\vec{B}$, $\vec{E}$ are the magnetic flux density and the electric flux density, respectively, $\vec{J}$ is the impressed current density (equal to zero), and ρ is density of source charges (equal to zero). Eqs. (A1) are used along with the constitutive relations, connecting the fields and the fluxes. We choose in accordance with [7],[12]

$$\begin{aligned}\frac{\partial \vec{D}}{\partial t} + \begin{bmatrix} d_x & 0 & 0 \\ 0 & d_y & 0 \\ 0 & 0 & d_z \end{bmatrix} \frac{\sigma}{\epsilon} \vec{D} &= \epsilon \frac{\partial \vec{E}}{\partial t} + \begin{bmatrix} e_x & 0 & 0 \\ 0 & e_y & 0 \\ 0 & 0 & e_z \end{bmatrix} \sigma \vec{E} \\ \frac{\partial \vec{B}}{\partial t} + \begin{bmatrix} d_x & 0 & 0 \\ 0 & d_y & 0 \\ 0 & 0 & d_z \end{bmatrix} \frac{\sigma^*}{\mu} \vec{B} &= \mu \frac{\partial \vec{H}}{\partial t} + \begin{bmatrix} e_x & 0 & 0 \\ 0 & e_y & 0 \\ 0 & 0 & e_z \end{bmatrix} \sigma^* \vec{H}\end{aligned}\tag{A2}$$

in order to describe the anisotropic loss mechanism with memory (dispersion). According to Ref. [11] this mechanism is equivalent to presence of dependent sources within the PML material. This mechanism is governed by two diagonal terms on the left-hand side

of Eqs. (A2). On the other hand, the diagonal terms on the right-hand side are identical to the relationships in a lossy isotropic medium.

The method of analysis of Eqs. (A1) and (A2) is based on the calculations of reflection coefficients for plane waves incident on the PML boundary, in the frequency domain, at different conditions [11]. A simplified 2D version (TM or TE) will be considered. If, for example, the PML face is *perpendicular to the z-axis*, then the choice

$$d_x = 0, d_y = 0, d_z = 1; \quad e_x = 1, e_y = 1, e_z = 0$$

$$\frac{\sigma^*}{\mu} = \frac{\sigma}{\varepsilon} \quad (A3)$$

solves the problem of reflectionless transmission [11],[12],[7]. For planes perpendicular to the x- and y-axes, the uniaxial parameters are permuted [12].

2 Frequency domain version of the anisotropic PML

The frequency-domain version of Eqs. (A1) and (A2) is often used. In particular, it is useful for the finite-element analysis. The frequency-domain version in terms of phasors (time dependence $\exp(j\omega t)$ is assumed) for the PML face that is *perpendicular to the z-axis* becomes

$$\vec{D} = \begin{bmatrix} 1 + \frac{\sigma}{j\omega\varepsilon} & 0 & 0 \\ 0 & 1 + \frac{\sigma}{j\omega\varepsilon} & 0 \\ 0 & 0 & \frac{1}{1 + \frac{\sigma}{j\omega\varepsilon}} \end{bmatrix} \varepsilon \vec{E} = \hat{A} \varepsilon \vec{E} \quad (A4)$$

$$\vec{B} = \hat{A} \mu \vec{E}$$

3 A 3D PML

The 3D PML, in FEM or FDTD, has 6 sides, 12 edges, and 8 corners. A typical picture of a FEM anisotropic PML created in Ansoft HFSS v. 12 is shown in Fig. A1 that follows. One thus can count total 26 bricks, labeled from PML_Box_1_1 to PML_Box_1_26 in Ansoft HFSS.

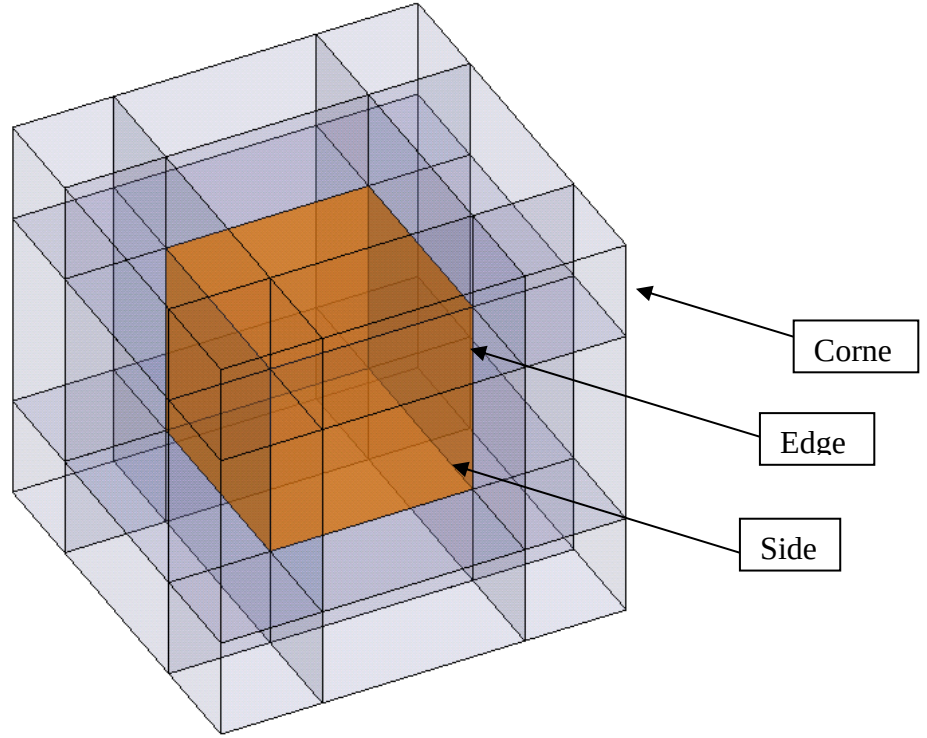


Fig. A1. Ansoft HFSS version of a 3D anisotropic PML box.

In order to assign parameters to sides, edges, and corners, we rewrite Eqs.(2) with

$$d_x = 0, d_y = 0, d_z = 1; \quad e_x = \overline{d_x}, e_y = \overline{d_y}, e_z = \overline{d_z}$$

$$\frac{\sigma^*}{\mu} = \frac{\sigma}{\varepsilon} \quad (\text{A5})$$

which allows us to determine dependent and independent parameters. Here, the overbar denotes logical NOT operation. The edge parameters are then those from both adjacent sides. For example, an edge between two PML sides being perpendicular to the z -axis and x -axis, respectively, has the set

$$d_x = 1, d_y = 0, d_z = 1; \quad e_x = 0, e_y = 1, e_z = 0$$

$$\frac{\sigma^*}{\mu} = \frac{\sigma}{\varepsilon} \quad (\text{A6})$$

etc. The corners parameters are all ones for d and for b , and all zeros otherwise.