

NEPAL—An $N^{1.5}$ Algorithm for Solving the Volume Integral Equation*

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I. Introduction

Recently, we have developed a recursive aggregate T-matrix algorithm (RATMA) to solve volume integral equation with reduced computational complexity compared to a direct solution via a matrix equation with Gaussian elimination [1-4]. The method has a computational complexity of N^2 for two dimensional scattering problems ($N^{7/3}$ for three dimensions) where N is the total number of unknowns needed to characterize the volume of the scatterer.

In this paper, we present an algorithm to solve volume integral equations with computational complexity of $N^{1.5}$ in two dimensions and N^2 in three dimensions. This is of reduced computational complexity compared to RATMA. We call this the nested equivalence principle algorithm (NEPAL).

The nested equivalence principle algorithm (NEPAL) is similar to many fast algorithm like FFT and the nested dissection ordering approach in finite element method [5]. The principal idea of this approach is to nest one algorithm within another so that a smaller problem is solved before a larger one. The size of the problem increases by a factor of 2^n at each stage where n is the dimension of the problem. In this manner, an N unknown problem is solved in $\log_n N$ steps.

II. The Volume Integral Equation

A volume integral equation has the form

$$\phi(\mathbf{r}) = \phi_{inc}(\mathbf{r}) + \overbrace{\int_V d\mathbf{r}' g_0(\mathbf{r} - \mathbf{r}') k_0^2 [\epsilon_r(\mathbf{r}') - 1] \phi(\mathbf{r}')}^{\phi_{scat}(\mathbf{r})} \quad (1)$$

or

$$\phi(\mathbf{r}) = \phi_{inc}(\mathbf{r}) + \underbrace{\int_V d\mathbf{r}' \nabla g_0(\mathbf{r} - \mathbf{r}') [\epsilon_r^{-1}(\mathbf{r}') - 1] \cdot \nabla' \phi(\mathbf{r}')}_{\phi_{scat}(\mathbf{r})} \quad (2)$$

where $g_0(\mathbf{r} - \mathbf{r}')$ is the Green's function.

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A usual practice in solving (1) and (2) is to discretize the scatterer into N discrete pieces and approximate the integrals in (1) with summations. Hence,

$$\phi_{sca}(\mathbf{r}) = \sum_{i=1}^N \psi^i(k_0, \mathbf{r}_i) \cdot \mathbf{b}_i, \quad (3)$$

where $\mathbf{r}_i = \mathbf{r} - \mathbf{r}'_i$, and $\psi^i(k_0, \mathbf{r}_i)$ is a row vector containing cylindrical harmonics in two dimensions and spherical harmonics in three dimensions. The column vector $\mathbf{b}_i$ contains the magnitudes of the harmonic expansions. For monopole subscatterers, $\mathbf{b}_i$ is a one-component vector corresponding to the monopole harmonics, while for dipole subscatterers, $\mathbf{b}_i$ is a three-component vector corresponding to three harmonics. The unknown $\mathbf{b}_i$ can be found by imposing a boundary condition on all the N subscatterers.

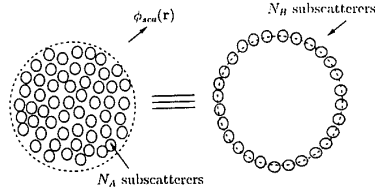


Figure 1. Huygens' equivalence principle helps to reduce the number of scattering centers.

III. The Huygens' Equivalence Principle

The Huygens' equivalence principle stated mathematically is

$$\phi_{sca}(\mathbf{r}) = \oint_S dS' \hat{n} \cdot [\phi_{sca}(\mathbf{r}') \nabla' g_0(\mathbf{r} - \mathbf{r}') - g_0(\mathbf{r} - \mathbf{r}') \nabla' \phi_{sca}(\mathbf{r}')]. \quad (4)$$

It can be used to shift the scattering centers of the subscatterers to the surface as shown in Figure 1. In this manner, the number of scattering centers can be reduced. If the number of volume subscatterers is $O(n)$, the number of surface subscatterers is then $O(n^{0.5})$ in two dimensions, and $O(n^{0.66})$ in three dimensions. This reduction in the number of scattering centers can be used to reduce the computational labor of solving for the scattering solution of a volume scatterer as illustrated by NEPAL.

IV. Nested Equivalence Principle Algorithm (NEPAL)

The nested equivalence principle algorithm (NEPAL) is similar in spirit to fast Fourier transform and nested dissection ordering in finite element

method. The idea is to divide the original problem into subgroups. The subgroups are then further divided into sub-subgroups until the lowest level of subgroups is reached. In two dimensions, the lowest level is in groups of four (see Figure 2) while in three dimensions, it is in groups of eight. (For 1D FFT, it is in groups of two.)

For instance, in two-dimensions, first, the scattering solutions for groups of four subscatterers are solved. Then the scattering solution from four groups each consisting of four subscatterers are solved. Hence, the scattering solution from a group of 16 subscatterers are known. Then, the Huygens' equivalence principle is used to replace 16 scattering centers with 12 scattering centers. Then the scattering solutions for four groups of scatterers each consisting of 12 subscatterers are solved. The process is repeated until the scattering solution for the entire scatterer is obtained.

Because of the use of Huygens' equivalence principle at each stage, the number of scatterers involved is reduced by roughly a factor of two when n is large. Hence, even if Gaussian elimination is used at each stage to solve for the scattering solution, reduction in computational labor is possible. It can be shown that when N gets sufficiently large, this is asymptotically an $O(N^{1.5})$ algorithm.

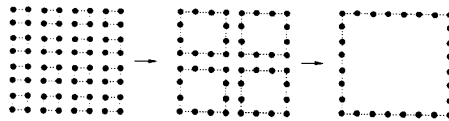


Figure 2. NEPAL—Nested equivalence principle algorithm.

V. Numerical Results

We have implemented NEPAL to solve volume integral equations (1) and (2) and compared the results with the directly solving the volume integral equations by the method of moments (MOM). The results are in good agreement for both E_z and H_z polarized waves as shown in the Figure 3 (a) and (b).

We have also studied the growth of computer time with the number of unknowns in the problem and found that NEPAL is very competitive with RATMA (see Figure 3(c)) in this preliminary investigation. It is certainly more efficient than MOM using Gaussian elimination. Moreover, the solution it provides is valid for all angles of incident unlike conjugate-type methods.

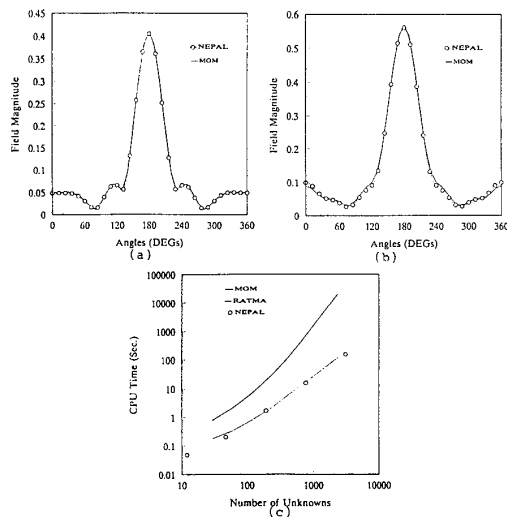


Figure 3. (a) Comparison with method of moments for E_z waves. (b) Comparison with method of moments for H_z waves. (c) Comparison of CPU time versus number of unknowns on a CRAY-2.

REFERENCES

- [1] W. C. Chew, "An N^2 algorithm for the multiple scattering solution of N scatterers," *Microwave Optical Tech. Lett.*, vol. 2, no. 11, pp. 380-383, 1989.
- [2] Y. M. Wang and W. C. Chew, "An efficient algorithm for solution of a scattering problem," *Microwave Optical Tech. Lett.*, vol. 3, no. 3, pp. 102-106, 1990.
- [3] W. C. Chew and Y. M. Wang, "A fast algorithm for solutions of a scattering problem using a recursive aggregate τ matrix method," *Microwave Optical Tech. Lett.*, vol. 3, no. 5, pp. 164-169, 1990.
- [4] W. C. Chew, *Waves and Field in Inhomogeneous Media*, Van Nostrand Reinhold, New York, 1990.
- [5] A. George, "Numerical experiments using dissection methods to solve $n \times n$ grid problems," *SIAM J. Numer. Anal.*, 14, pp. 161-179, 1977.