

Splines and Their Reciprocal-Bases in Volume-Integral Equations

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Abstract—We briefly outline the use of higher-order splines and their reciprocal-bases in discretizing the volume-integral equations of electromagnetics. The discretization is carried out by means of the method of moments, in which the expansion functions are the higher-order splines, and the testing functions are the corresponding reciprocal-basis functions. These functions satisfy an orthogonality condition with respect to the spline expansion functions. Thus, the method is not Galerkin, but the structure of the resulting equations is quite regular, nevertheless. The theory is applied to the volume-integral equations for the unknown current density, or unknown electric field, within a scattering body, and to the equations for eddy-current non-destructive evaluation. Numerical techniques for computing the matrix elements are also given.

INTRODUCTION

VOLUME-INTEGRAL equations are especially useful in solving electromagnetics problems involving “penetrable bodies,” such as arise in eddy-current non-destructive evaluation. Typically, these equations are discretized by means of the method of moments, in which the unknown fields are expanded in one class of functions, and the resulting equations are then tested by using another class of functions. When the expansion and testing functions are identical, the method is called the Galerkin variant of the method of moments. In any case, it is important to choose these functions so that they possess the requisite continuity for the actual fields, and so that the resulting discretized equations have attractive properties, such as relatively few unknowns, the matrices can be relatively easily computed, etc.

Quite often, one uses pulse functions for expansion and testing (which reverts to Galerkin’s method), or pulse functions for expansion and delta functions for testing (which is called “point-matching” or “collocation”). To produce smoother solutions, however, it may be necessary to use functions that are smoother than pulses, while at the same time using functions for testing that are in some sense optimum for the expansion functions. We explore the possibility of using higher-order splines for expansion, and the reciprocal-bases for these splines for testing. We are concerned only with the frequency do-

main, in this paper, but we have applied the theory of splines and their reciprocal-bases to time-domain volume-integral equations, as well [1].

I. A FREQUENCY-DOMAIN VOLUME-INTEGRAL EQUATION

We are interested in computing the currents, or electric fields, induced within a scattering body when a known field is incident upon the body. The incident field could be a plane wave coming from infinity, or the field produced by a probe coil in the vicinity of the body, as in nondestructive evaluation. We assume that the scatterer lies above an infinite, plane-parallel region, which could be either a half-space or a slab of finite thickness. In fact, we have developed the frequency-domain theory to include highly anisotropic slabs.

We start our development in the frequency-domain, and note that the solution of Maxwell’s equations is given by the following volume-integral relation:

$$E^{(i)}(r, \omega) = E(r, \omega) + \int_{\text{scat}} \mathcal{G}(r|r', \omega) \cdot J(r', \omega) dr', \quad (1)$$

where $\mathcal{G}$ is the dyadic Green’s function for the problem, $E^{(i)}$ is the incident field that exists in the absence of the scattering body, and J is the electric current coupled into the body, and which accounts for the scattered field. We are assuming that there are no magnetic effects, so that the only unknowns are the electric field and current induced into the scattering body. Coupled electric and magnetic field problems can be easily accommodated by introducing the appropriate dyadic Green’s function. It is important to note that (1) is a linear relation between J and E , because it solves Maxwell’s equations, even if the constitutive relation between J and E is nonlinear.

The Green’s dyad can be written in the form of a two-dimensional Fourier transform,

$$\begin{aligned} \mathcal{G}(x - x', y - y', z, z', \omega) \\ = \iint_{-\infty}^{\infty} \tilde{\mathcal{G}}(k_x, k_y, z, z'; \omega) e^{-j[k_x(x - x') + k_y(y - y')]} dk_x dk_y, \end{aligned} \quad (2)$$

and when this is substituted into (1) the integrals over x' and y' become two-dimensional spatial Fourier transforms.

The Green’s function consists of two parts, a free-space

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part, which ignores the presence of the slab (and, therefore, could be called the incident part) and a scattered (or reflected) part. The free-space part, in turn, consists of two parts, $\tilde{G}^{(0)}$, and a "depolarizing" part, $\tilde{G}^{(d)}$. $\tilde{G}^{(0)}$ is given by

$$\tilde{G}^{(0)}(k_x, k_y; z, z') = \frac{-j\omega\mu_0}{4\pi^2} \begin{bmatrix} 1 - k_x^2/k_0^2 & -k_x k_y/k_0^2 & \pm jk_x \lambda_0/k_0^2 \\ -k_x k_y/k_0^2 & 1 - k_y^2/k_0^2 & \pm jk_y \lambda_0/k_0^2 \\ \pm jk_x \lambda_0/k_0^2 & \pm jk_y \lambda_0/k_0^2 & 1 + \lambda_0^2/k_0^2 \end{bmatrix} \frac{e^{-\lambda_0|z-z'|}}{2\lambda_0}, \quad (3)$$

where the (+) sign goes with $z > z'$, (-) with $z < z'$, and $\lambda_0 = (k_x^2 + k_y^2 - k_0^2)^{1/2}$.

The depolarizing part consists of the single term

$$\tilde{G}^{(d)}(k_x, k_y; z, z') = j \frac{\delta(z - z')}{4\pi^2 \omega \epsilon_0} a_z a_z, \quad (4)$$

and the reflected Green's function has the structure

$$e^-(z + z') \tilde{G}^{(r)}(k_x, k_y; z, z') = \frac{1}{4\pi^2} \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} \quad (5)$$

where the matrix elements are independent of (z, z') .

Equation (1) is the basic frequency-domain volume-integral equation. We are going to discretize it by using higher-order splines and their reciprocal-basis functions. The discussion of higher-order splines is based on the theory presented in Aubin's book [2], chapters 4 and 5. The computation of the reciprocal-basis system for splines is outlined in Section 2-5 of Chapter 4 of [2]. For more information on the idea of reciprocal-bases, see L. E. Franks' book [3]. The development is for functions of one variable, but the extension to multivariables is easily carried out by taking tensor products of single variable functions.

Though we use Aubin's notation, the application of the theory of higher-order splines and reciprocal-bases to the discretization of the integral equations of electromagnetics, as developed here, is novel. This paper considerably extends the work presented in [4].

II. SPLINES BASED UPON HIGHER-ORDER CONVOLUTIONS OF THE UNIT PULSE

Consider a regular one-dimensional grid, whose spacing is h . Relative to this grid we define $\pi_1(x)$ to be the unit pulse

$$\pi_1(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad (6)$$

and $\pi_{m+1}(x)$ to be the m th-order convolution of $\pi_1(x)$. The $\pi_{m+1}(x)$ are shown in Fig. 1 for $m = 0, 1, 2, 3$.

The higher-order convolutions of the unit pulse can be written in terms of the piecewise polynomials, $\alpha_m^k(x)$, as

$$\pi_{m+1}(x) = \sum_{k=0}^m \alpha_m^k(x - k) \pi_1(x - k),$$

and the $\alpha_m^k(x) = \sum_{j=0}^m a_m(k, j) x^j / j!$ are tabulated in Table I. The coefficients $a_m(k, j)$ satisfy the recursion relation

$$a_m(k, j) = a_{m-1}(k, j-1) - a_{m-1}(k-1, j-1), \quad 1 \leq k \leq m-1, 1 \leq j \leq m. \quad (7)$$

Numerical values for $a_m(k, j)$, for $1 \leq m \leq 3$, can be easily inferred from Table I. For example, $a_3(k, j)$ is given by the following matrix,

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & 1/2 & 1 & -3 \\ 4/6 & 0 & -2 & 3 \\ 1/6 & -1/2 & 1 & -1 \end{bmatrix}.$$

It will be useful to list the one-dimensional Fourier transforms of the translates of these functions:

$$\begin{aligned} \tilde{\pi}_1(x/h - l) &= h e^{jk(l+1/2)h} \left(\frac{\sin(kh/2)}{kh/2} \right) \\ \tilde{\pi}_2(x/h - l) &= h e^{jk(l+1)h} \left(\frac{\sin(kh/2)}{kh/2} \right)^2 \\ \tilde{\pi}_3(x/h - l) &= h e^{jk(l+3/2)h} \left(\frac{\sin(kh/2)}{kh/2} \right)^3 \\ \tilde{\pi}_4(x/h - l) &= h e^{jk(l+2)h} \left(\frac{\sin(kh/2)}{kh/2} \right)^4. \end{aligned} \quad (8)$$

k is the transform variable. Note the relationship between the Fourier transforms:

$$\frac{\tilde{\pi}_n}{\tilde{\pi}_{n-1}} = e^{jk h/2} \left(\frac{\sin(kh/2)}{kh/2} \right). \quad (9)$$

III. RECIPROCAL-BASES FOR $\pi_{m+1}(x)$

Define $\lambda_m(x)$ to be a function of compact support that will generate the reciprocal-basis system, just as $\pi_{m+1}(x)$ generates the basis system.

We want

$$(h \pi_{m+1}^j, \lambda_{m,h}^k) = \delta_{jk}, \quad (10)$$

where the parentheses denote an integral; $\delta_{jk} = 1$, if $j = k$, $= 0$, if $j \neq k$, and

$$\pi_{m+1,h}^j = \frac{1}{h} \pi_{m+1}(x/h - j), \quad \lambda_{m,h}^k = \frac{1}{h} \lambda_m(x/h - k). \quad (11)$$

Upon substituting (11) into (10), we get

$$\frac{1}{h} \int \pi_{m+1}(x/h - j) \lambda_m(x/h - k) dx = \delta_{jk}. \quad (12)$$

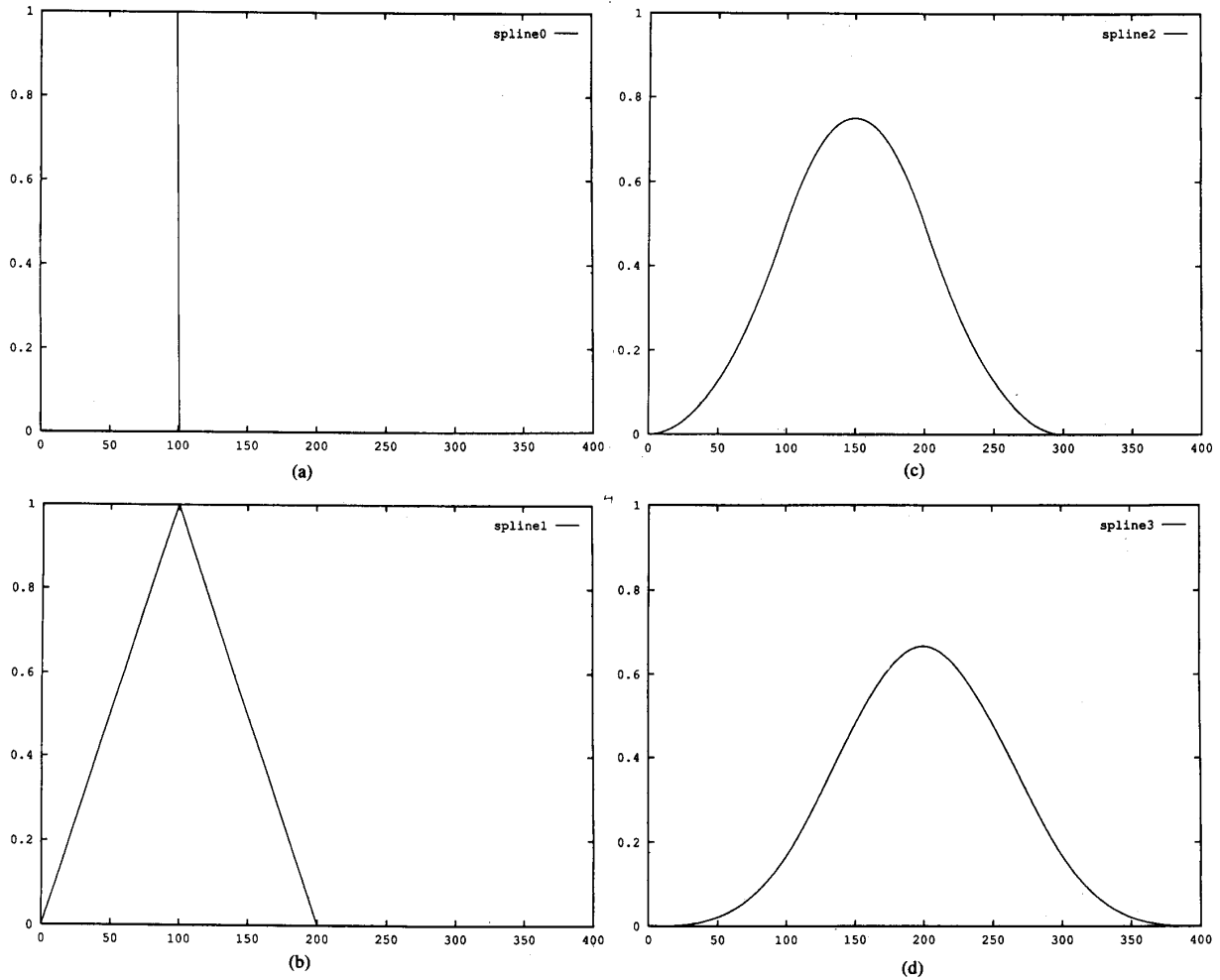
Fig. 1. $\pi_{m+1}(x)$: (a) $m = 0$; (b) $m = 1$; (c) $m = 2$; (d) $m = 3$.

TABLE I
PIECEWISE POLYNOMIALS, $\alpha_m^k(x)$, IN $\pi_{m+1}(x) = \sum_{k=0}^m \alpha_m^k(x-k) \pi_1(x-k)$

Order (m)	$\alpha_m^k(x) = \sum_{j=0}^m a_m(k, j) x^j / j!$
$m = 0$	$\alpha_0^0(x) = 1$
$m = 1$	$\alpha_1^0(x) = x$ $\alpha_1^1(x) = 1 - x$
$m = 2$	$\alpha_2^0(x) = x^2/2$ $\alpha_2^1(x) = 1/2 + x - x^2$ $\alpha_2^2(x) = 1/2 - x + x^2/2$
$m = 3$	$\alpha_3^0(x) = x^3/6$ $\alpha_3^1(x) = 1/6 + x/2 + x^2/2 - x^3/2$ $\alpha_3^2(x) = 4/6 - x^2 + x^3/2$ $\alpha_3^3(x) = 1/6 - x/2 + x^2/2 - x^3/6$

Now let

$$\lambda_m(x) = \pi_1(x) \sum_{0 \leq l \leq m} b_l^{(m)} \frac{x^l}{l!}. \quad (13)$$

Then

$$\lambda_m(x/h - k) = \pi_1(x/h - k) \sum_{0 \leq l \leq m} b_l^{(m)} \frac{(x/h - k)^l}{l!}, \quad (14)$$

and we get

$$\sum_{0 \leq l \leq m} b_l^{(m)} \frac{1}{h} \int_{kh}^{(k+1)h} \pi_{m+1}(x/h - j) \frac{(x/h - k)^l}{l!} dx = \delta_{jk}. \quad (15)$$

Now let

$$x/h - k = z \Rightarrow dx/h = dz, \quad x/h - j = z + k - j.$$

Then

$$\sum_{0 \leq l \leq m} b_l^{(m)} \int_0^1 \pi_{m+1}(z + k - j) \frac{z^l}{l!} dz = \delta_{jk}. \quad (16)$$

We must determine the restriction of $\pi_{m+1}(z + k - j)$ to the interval, $(0, 1)$:

$$\pi_{m+1}(z) = \sum_{n=0}^m \alpha_m^n(z - n) \pi_1(z - n). \quad (17)$$

This implies that

$$\begin{aligned} \pi_{m+1}(z + k - j) \\ = \sum_{n=0}^m \alpha_m^n(z + k - (j + n)) \pi_1(z + k - (j + n)). \end{aligned} \quad (18)$$

Hence, because of the presence of the pulse function, $\pi_1(z + k - (j + n))$, the term that survives in the interval (0, 1) is $k = j + n$; i.e.,

$$\pi_{m+1}(z) = \alpha_m^{(k-j)}(z), \quad \text{for } 0 \leq z \leq 1. \quad (19)$$

$$\begin{aligned} h_{00} &= 1/24, & h_{01} &= 1/30, & h_{02} &= 1/72, & h_{03} &= 1/252 \\ h_{10} &= 11/24, & h_{11} &= 11/40, & h_{12} &= 71/720, & h_{13} &= 43/1680 \\ h_{20} &= 11/24, & h_{21} &= 11/60, & h_{22} &= 19/360, & h_{23} &= 1/84 \\ h_{30} &= 1/24, & h_{31} &= 1/120, & h_{32} &= 1/720, & h_{33} &= 1/5040. \end{aligned} \quad (25)$$

Thus,

$$\sum_{0 \leq l \leq m} b_l^{(m)} \int_0^1 \alpha_m^{(k-j)}(z) \frac{z^l}{l!} dz = \delta_{jk}, \quad (20)$$

or,

$$\sum_{0 \leq l \leq m} b_l^{(m)} \int_0^1 \alpha_m^n(z) \frac{z^l}{l!} dz = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } 1 \leq n \leq m. \end{cases} \quad (21)$$

We apply these results to the case of cubic splines, for which $m = 3$. From Table I we have

$$\begin{aligned} \alpha_3^0(x) &= \frac{x^3}{6} \\ \alpha_3^1(x) &= \frac{1}{6} + \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{2} \\ \alpha_3^2(x) &= \frac{4}{6} - x^2 + \frac{x^3}{2} \\ \alpha_3^3(x) &= \frac{1}{6} - \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{6}. \end{aligned} \quad (22)$$

Thus, we can compute the matrix elements

$$\begin{aligned} h_{kj}^3 &= \sum_{l=0}^3 a_3(k, l) \int_0^1 \frac{x^{j+l}}{j! l!} dx \\ &= \sum_{l=0}^3 \frac{a_3(k, l)}{(j+k+1)j! l!}, \quad j, k = 0, 1, 2, 3. \end{aligned} \quad (23)$$

From this expression and the expression for the $a_3(k, i)$

given above, we derive the following expressions:

$$\begin{aligned} h_{0j}^3 &= \frac{1}{j!} \left[\frac{1}{6(j+4)} \right] \\ h_{1j}^3 &= \frac{1}{j!} \left[\frac{1/6}{j+1} + \frac{1/2}{j+2} + \frac{1}{2(j+3)} - \frac{3}{6(j+4)} \right] \\ h_{2j}^3 &= \frac{1}{j!} \left[\frac{4/6}{j+1} - \frac{2}{2(j+3)} + \frac{3}{6(j+4)} \right] \\ h_{3j}^3 &= \frac{1}{j!} \left[\frac{1/6}{j+1} - \frac{1/2}{j+2} + \frac{1}{2(j+3)} - \frac{1}{6(j+4)} \right]. \end{aligned} \quad (24)$$

The numerical values of the matrix elements are

This yields the matrix equation

$$\begin{bmatrix} 1/24 & 1/30 & 1/72 & 1/252 \\ 11/24 & 11/40 & 71/720 & 43/1680 \\ 11/24 & 11/60 & 19/360 & 1/84 \\ 1/24 & 1/120 & 1/720 & 1/5040 \end{bmatrix} \begin{bmatrix} b_0^{(3)} \\ b_1^{(3)} \\ b_2^{(3)} \\ b_3^{(3)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (26)$$

The solution of (26) is

$$\begin{aligned} b_0^{(3)} &= -184, & b_1^{(3)} &= 2460, & b_2^{(3)} &= -13200, \\ b_3^{(3)} &= 27720, \end{aligned} \quad (27)$$

which, from (13), means that the reciprocal-basis for the cubic splines is generated from the function

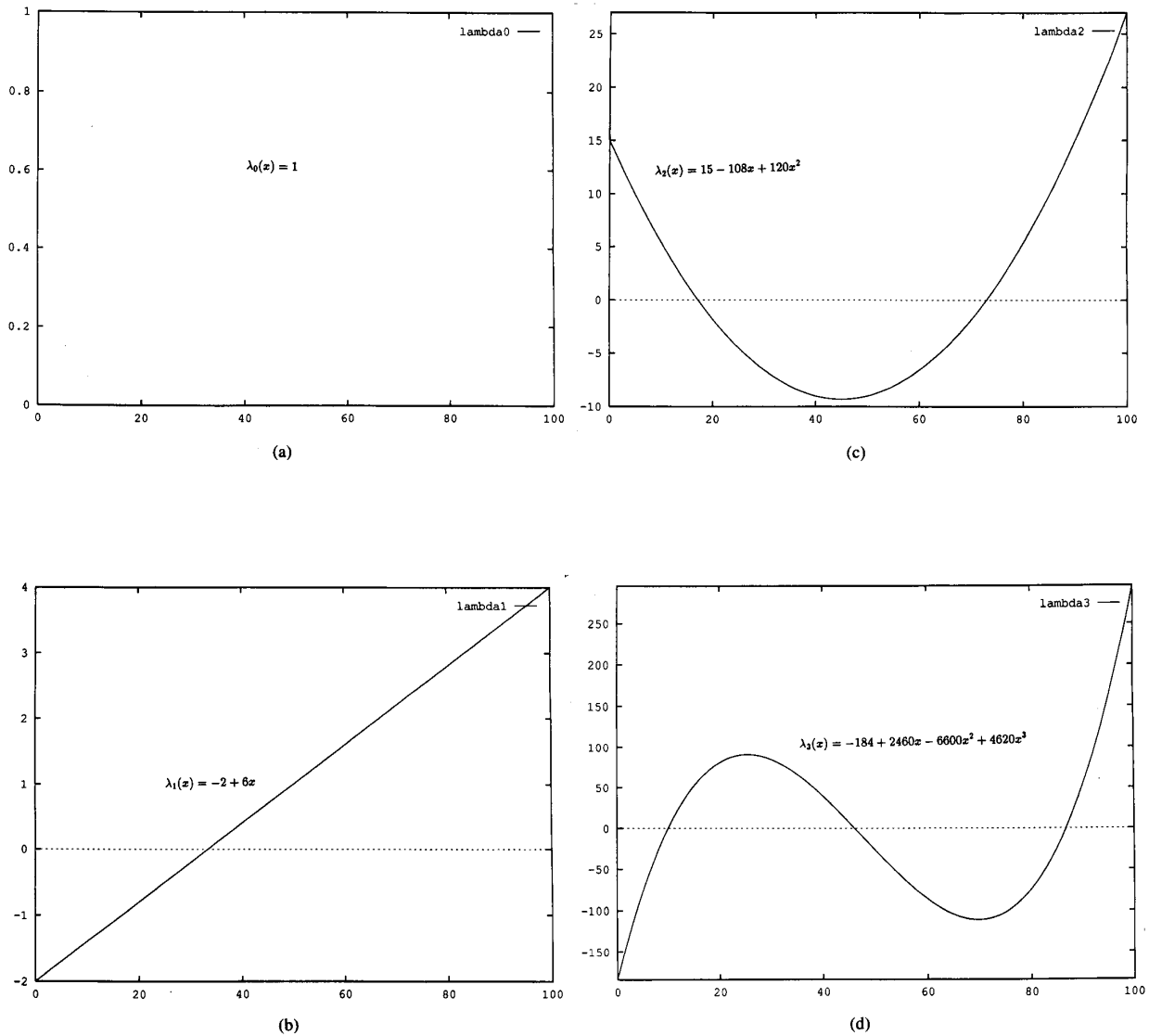
$$\lambda_3(x) = \pi_1(x)(-184 + 2460x - 6600x^2 + 4620x^3), \quad (28)$$

and its translates, as in (14).

A similar analysis for splines of lower order produces the following results:

$$\begin{aligned} \lambda_2(x) &= \pi_1(x)(15 - 108x + 120x^2) \\ \lambda_1(x) &= \pi_1(x)(-2 + 6x) \\ \lambda_0(x) &= \pi_1(x) \end{aligned} \quad (29)$$

These functions are plotted in Fig. 2.

Fig. 2. $\lambda_m(x)$: (a) $m = 0$; (b) $m = 1$; (c) $m = 2$; (d) $m = 3$.

Aubin [2, page 124] shows that the integral of $\lambda_m(x)$ is unity; a check of the results above shows that this is satisfied. Hence, we have the reciprocal-basis for the splines of order 0 to 3.

The utility of the reciprocal-basis stems from (12), in that if we expand an arbitrary function, $F(x)$, in terms of $\pi_{m+1}(x)$

$$F(x) = \sum_j I_j \pi_{m+1}(x/h - j), \quad (30)$$

then the expansion coefficients can be found immediately

$$I_j = \frac{1}{h} \int F(x) \lambda_m(x/h - j) dx. \quad (31)$$

This is useful in discretizing functional equations, and in the interpolation and smoothing of tabular data, as well.

We list the Fourier transforms of $\lambda_m(x/h - l)$:

$$\begin{aligned} \tilde{\lambda}_0(x/h - l) &= h e^{i l} \tilde{\lambda}_0(\xi) \\ \tilde{\lambda}_1(x/h - l) &= h e^{i l} \tilde{\lambda}_1(\xi) \\ \tilde{\lambda}_2(x/h - l) &= h e^{i l} \tilde{\lambda}_2(\xi) \\ \tilde{\lambda}_3(x/h - l) &= h e^{i l} \tilde{\lambda}_3(\xi), \end{aligned} \quad (32)$$

where $\xi = j h k$ (k being the transform variable), and the

$\tilde{\lambda}$'s are given by

$$\begin{aligned}\tilde{\lambda}_0(\xi) &= \frac{e^\xi - 1}{\xi} \\ \tilde{\lambda}_1(\xi) &= \frac{e^\xi(4\xi - 6) + 2\xi + 6}{\xi^2} \\ \tilde{\lambda}_2(\xi) &= \frac{e^\xi(27\xi^2 - 132\xi + 240) - 15\xi^2 - 108\xi - 240}{\xi^3} \\ \tilde{\lambda}_3(\xi) &= \frac{e^\xi(296\xi^3 - 3120\xi^2 + 14520\xi - 27720) + 184\xi^3 + 2460\xi^2 + 13200\xi + 27720}{\xi^4}.\end{aligned}\quad (33)$$

Note that $\tilde{\lambda}_m(\xi) \rightarrow 1$, as $\xi \rightarrow 0$, which is in agreement with the statement that the integral of $\lambda_m(x)$ is unity. The results of (32) and (33) will be useful when using the reciprocal-bases to discretize integral equations.

In (30) we expanded an arbitrary function, $F(x)$, in terms of the m th-order convolution of the unit pulse. Now we want to write the restriction of the expansion to each interval as

$$F_m(x) = \sum_j \pi_{1j}(x) \sum_{k=0}^m I_{j-k} \alpha_m^k(x/h - j), \quad (34)$$

where $\pi_{1j}(x)$ is the unit pulse over $(jh, (j+1)h)$. The result is

$$\begin{aligned}F_0(x) &= \sum_j \pi_{1j}(x) [I_j] \\ F_1(x) &= \sum_j \pi_{1j}(x) [I_j - 1 + (I_j - I_{j-1})(x/h - j)] \\ F_2(x) &= \sum_j \pi_{1j}(x) [1/2(I_{j-1} + I_{j-2}) \\ &\quad + (I_{j-1} - I_{j-2})(x/h - j) \\ &\quad + 1/2(I_j - 2I_{j-1} + I_{j-2})(x/h - j)^2] \\ F_3(x) &= \sum_j \pi_{1j}(x) [1/6(I_{j-1} + 4I_{j-2} + I_{j-3}) \\ &\quad + 1/2(I_{j-1} - I_{j-3})(x/h - j) \\ &\quad + 1/2(I_{j-1} - 2I_{j-2} + I_{j-3})(x/h - j)^2 \\ &\quad + 1/6(I_j - 3I_{j-1} + 3I_{j-2} - I_{j-3})(x/h - j)^3].\end{aligned}\quad (35)$$

These results are useful when carrying out mathematical operations, such as differentiation and integration, on the expansion.

IV. DISCRETIZING INTEGRAL EQUATIONS WITH HIGHER-ORDER SPLINES

In this section we will consider linear problems for which $\mathbf{J} = \sigma \mathbf{E}$ holds for the scatterer. If we multiply (1) by $\sigma(\mathbf{r})$, we get the basic integral equation for the current

within the scatterer:

$$\mathbf{J}^{(i)}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) + \sigma(\mathbf{r}) \int_{\text{scat}} \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') d\mathbf{r}', \quad (36)$$

where the Green's dyad is given in (2). Thus

$$\begin{aligned}\mathbf{J}^{(i)}(\mathbf{r}) &= \mathbf{J}(\mathbf{r}) + \sigma(\mathbf{r}) \int dz' \int \int_{-\infty}^{\infty} \tilde{\mathbf{G}}(k_x, k_y; z, z') \\ &\quad \cdot \tilde{\mathbf{J}}(k_x, k_y; z') e^{-j[k_x(x-x') + k_y(y-y')]} dk_x dk_y.\end{aligned}\quad (37)$$

Introduce a regular, three-dimensional grid, whose spacing is δx , δy , and δz , and expand $\mathbf{J}(\mathbf{r})$ as the three-dimensional tensor-product of one-dimensional splines of arbitrary order, n , defined on this grid:

$$\mathbf{J}(\mathbf{r}) = \sum_l \sum_m \sum_k \mathbf{I}_{lmk} F_l^{(n)}(x/\delta x) F_m^{(n)}(y/\delta y) F_k^{(n)}(z/\delta z), \quad (38)$$

so that

$$\tilde{\mathbf{J}}(k_x, k_y; z) = \sum_l \sum_m \sum_k \mathbf{I}_{lmk} \tilde{F}_l^{(n)}(k_x) \tilde{F}_m^{(n)}(k_y) F_k^{(n)}(z/\delta z). \quad (39)$$

Then,

$$\begin{aligned}\mathbf{J}^{(i)}(\mathbf{r}) &= \sum_l \sum_m \sum_k \mathbf{I}_{lmk} F_l^{(n)}(x/\delta x) F_m^{(n)}(y/\delta y) F_k^{(n)}(z/\delta z) \\ &\quad + \sum_L \sum_M \sum_K \sigma(\mathbf{r}) \int \int_{-\infty}^{\infty} e^{-j[k_L x + k_M y]} \\ &\quad \cdot \tilde{F}_L^{(n)}(k_x) \tilde{F}_M^{(n)}(k_y) \\ &\quad \cdot \left\{ \int dz' \tilde{\mathbf{G}}(k_x, k_y; z, z') \cdot \mathbf{I}_{LMK} F_K^{(n)}(z'/\delta z) \right\} \\ &\quad \cdot dk_x dk_y.\end{aligned}\quad (40)$$

Our notation is $F_l^{(n)}(x/\delta x) = \pi_{n+1}(x/\delta x - l)$.

Let $\{\lambda_l^{(n)}(x/\delta x)\}$ be the reciprocal-basis to $\{F_l^{(n)}(x/\delta x)\}$. This means that

$$\int \lambda_l^{(n)}(x/\delta x) F_l^{(n)}(x/\delta x) dx/\delta x = \delta_{ll'}. \quad (41)$$

We are going to take moments of (40) with respect to

the reciprocal-basis. Hence, we first multiply (40) by $\lambda_l^{(n)}(x/\delta x)\lambda_m^{(n)}(y/\delta y)\lambda_k^{(n)}(z/\delta z)$, which yields:

$$\begin{aligned} & \lambda_l^{(n)}(x/\delta x)\lambda_m^{(n)}(y/\delta y)\lambda_k^{(n)}(z/\delta z)J^{(i)}(r) \\ &= \sum_l \sum_m \sum_k I_{lmk} \lambda_l^{(n)}(x/\delta x)\lambda_m^{(n)}(y/\delta y)\lambda_k^{(n)}(z/\delta z) \\ & \quad \cdot F_l^{(n)}(x/\delta x)F_m^{(n)}(y/\delta y)F_k^{(n)}(z/\delta z) \\ & \quad + \sum_L \sum_M \sum_K \lambda_l^{(n)}(x/\delta x)\lambda_m^{(n)}(y/\delta y)\lambda_k^{(n)}(z/\delta z)\sigma(r) \\ & \quad \cdot \int_{-\infty}^{\infty} e^{-j(k_x x + k_y y)} \tilde{F}_L^{(n)}(k_x) \tilde{F}_M^{(n)}(k_y) \\ & \quad \cdot \left\{ \int dz' \tilde{G}(k_x, k_y; z, z') \cdot I_{LMK} F_K^{(n)}(z'/\delta z) \right\} \\ & \quad \cdot dk_x dk_y. \end{aligned} \quad (42)$$

Now we postulate the following form:

$$\begin{aligned} & \lambda_l^{(n)}(x/\delta x)\lambda_m^{(n)}(y/\delta y)\lambda_k^{(n)}(z/\delta z)\sigma(r) \\ &= F_l^{(n)}(x/\delta x)F_m^{(n)}(y/\delta y)F_k^{(n)}(z/\delta z)\sigma_{l'm'k'}, \end{aligned} \quad (43)$$

where $\sigma_{l'm'k'}$ is a constant given by

$$\begin{aligned} \sigma_{l'm'k'} &= \frac{1}{\delta x \delta y \delta z} \int (\lambda_l^{(n)}(x/\delta x))^2 (\lambda_m^{(n)}(y/\delta y))^2 \\ & \quad \cdot (\lambda_k^{(n)}(z/\delta z))^2 \sigma(r) dx dy dz. \end{aligned} \quad (44)$$

Hence, $\sigma_{l'm'k'}$ is the weighted average (with a positive weighting function) of σ over the $l'm'k'$ -th cell. This is exactly what we got when we used pulse-functions for expansion and testing in our previous work [5], [6]. The form postulated in (43) sort of smears $\sigma(r)$ over several cells in the x , y , and z directions. The constant given in (44) is optimally defined to produce this smearing.

Now, we substitute (43) back into (42) and integrate with respect to (x, y, z) ; making use of (41), we have

$$\begin{aligned} I_{l'm'k'}^{(i)} &= I_{l'm'k'} + \sigma_{l'm'k'} \sum_L \sum_M \sum_K \left[\int_{-\infty}^{\infty} \tilde{F}_L^{*(n)}(k_x) \right. \\ & \quad \cdot \tilde{F}_L^{(n)}(k_x) \tilde{F}_M^{*(n)}(k_y) \tilde{F}_M^{(n)}(k_y) \\ & \quad \cdot \left\{ \int dz \int dz' \tilde{G}(k_x, k_y; z, z') F_K^{(n)}(z'/\delta z) \right. \\ & \quad \cdot F_K^{(n)}(z'/\delta z) \left. \right\} dk_x dk_y \left. \right] \cdot I_{LMK}. \end{aligned} \quad (45)$$

Upon rearranging this becomes

$$I_{lmk}^{(i)} = I_{lmk} + \sigma_{lmk} \sum_L \sum_M \sum_K G_{kK}(l-L, m-M) \cdot I_{LMK}, \quad (46)$$

where,

$$\begin{aligned} I_{lmk}^{(i)} &= \frac{1}{\delta x \delta y \delta z} \int \int \int J^{(i)}(r) \lambda_l^{(n)}(x/\delta x) \lambda_m^{(n)}(y/\delta y) \\ & \quad \cdot \lambda_k^{(n)}(z/\delta z) dr \end{aligned}$$

$$\begin{aligned} &= \frac{\sigma_{lmk}}{\delta x \delta y \delta z} \int \int \int F_l^{(n)}(x/\delta x) F_m^{(n)}(y/\delta y) F_k^{(n)}(z/\delta z) E^{(i)}(r) dr \\ & \quad \cdot (y/\delta y) F_k^{(n)}(z/\delta z) E^{(i)}(r) dr \end{aligned}$$

$$\begin{aligned} \sigma_{lmk} &= \frac{1}{\delta x \delta y \delta z} \int \int \int \sigma(r) \left[\lambda_l^{(n)}(x/\delta x) \right]^2 \\ & \quad \cdot \left[\lambda_m^{(n)}(y/\delta y) \right]^2 \left[\lambda_k^{(n)}(z/\delta z) \right]^2 dr \end{aligned}$$

$$\begin{aligned} G_{kK}(l-L, m-M) &= \int_{-\infty}^{\infty} \tilde{F}_L^{*(n)}(k_x) \tilde{F}_L^{(n)}(k_x) \tilde{F}_M^{*(n)}(k_y) \\ & \quad \cdot \tilde{F}_M^{(n)}(k_y) \tilde{\Gamma}_{kK}(k_x, k_y) dk_x dk_y \end{aligned}$$

$$\begin{aligned} \tilde{\Gamma}_{kK}(k_x, k_y) &= \int dz \int dz' \tilde{G}(k_x, k_y; z, z') \\ & \quad \cdot F_k^{(n)}(z/\delta z) F_K^{(n)}(z'/\delta z). \end{aligned} \quad (47)$$

Note that

$$\tilde{F}_L^{*(n)}(k_x) \tilde{F}_L^{(n)}(k_x) = e^{jk_x(L-l)} \left(\frac{\sin(k_x/2)}{k_x/2} \right)^{2(n+1)},$$

which is a function of the difference of the two arguments, l and L . If $n = 0$, we get the expansion and testing in pulse-functions that were used in [5] and [6].

Now let us consider the basic integral equation for the electric field, rather than the current:

$$E^{(i)}(r) = E(r) + \int_{\text{scat}} G(r; r') \cdot \sigma(r') dr', \quad (48)$$

which also follows from (1). This integral equation is used in rigorously solving electromagnetic inverse problems in nondestructive evaluation [7], [8]. Expand $E(r)$ in splines of order n :

$$E(r) = \sum_l \sum_m \sum_k E_{lmk} F_l^{(n)}(x/\delta x) F_m^{(n)}(y/\delta y) F_k^{(n)}(z/\delta z). \quad (49)$$

Then, (48) becomes

$$\begin{aligned} E^{(i)}(r) &= \sum_l \sum_m \sum_k E_{lmk} F_l^{(n)}(x/\delta x) F_m^{(n)}(y/\delta y) F_k^{(n)}(z/\delta z) \\ & \quad + \sum_L \sum_M \sum_K \left[\int_{\text{scat}} \sigma(r') F_L^{(n)}(x'/\delta x) F_M^{(n)}(y'/\delta y) \right. \\ & \quad \cdot F_K^{(n)}(z'/\delta z) G(r; r') dx' dy' dz' \left. \right] E_{LMK} \end{aligned} \quad (50)$$

Now we postulate the form:

$$\begin{aligned} & \sigma(r') F_L^{(n)}(x'/\delta x) F_M^{(n)}(y'/\delta y) F_K^{(n)}(z'/\delta z) \\ &= \sigma_{LMK} \lambda_L^{(n)}(x'/\delta x) \lambda_M^{(n)}(y'/\delta y) \lambda_K^{(n)}(z'/\delta z), \end{aligned} \quad (51)$$

where σ_{LMK} is a constant given by

$$\begin{aligned} \sigma_{LMK} &= \frac{1}{\delta x \delta y \delta z} \int (F_L^{(n)}(x/\delta x))^2 (F_M^{(n)}(y/\delta y))^2 \\ & \quad \cdot (F_K^{(n)}(z/\delta z))^2 \sigma(r) dx dy dz. \end{aligned} \quad (52)$$

Equation (51) is sort of the "transpose" of (43), while (52) corresponds to (44).

Hence, upon substituting (51) into (50), we get

$$\begin{aligned} E^{(i)}(r) &= \sum_l \sum_m \sum_k E_{lmk} F_l^{(n)}(x/\delta x) F_m^{(n)}(y/\delta y) F_k^{(n)}(z/\delta z) \\ &\quad + \sum_L \sum_M \sum_K \left[\int_{\text{scat}} G(r; r') \lambda_L^{(n)}(x'/\delta x) \lambda_M^{(n)}(y'/\delta y) \right. \\ &\quad \left. \cdot \lambda_K^{(n)}(z'/\delta z) dx' dy' dz' \right] \cdot \sigma_{LMK} E_{LMK} \\ &= \sum_l \sum_m \sum_k E_{lmk} F_l^{(n)}(x/\delta x) F_m^{(n)}(y/\delta y) F_k^{(n)}(z/\delta z) \\ &\quad + \sum_L \sum_M \sum_K \left[\int \int_{-\infty}^{\infty} e^{-j(k_x x + k_y y)} \tilde{\lambda}_L^{(n)}(k_x) \tilde{\lambda}_M^{(n)}(k_y) \right. \\ &\quad \left. \cdot \left\{ \int dz' \tilde{G}(k_x, k_y; z, z') \lambda_K^{(n)}(z'/\delta z) \right\} dk_x dk_y \right] \\ &\quad \cdot \sigma_{LMK} E_{LMK}. \end{aligned} \quad (53)$$

Taking moments of (53) with respect to $\lambda_l^{(n)}(x/\delta x) \lambda_m^{(n)}(y/\delta y) \lambda_k^{(n)}(z/\delta z)$ yields

$$\begin{aligned} E_{lmk}^{(i)} &= E_{lmk} + \sum_L \sum_M \sum_K \left[\int \int_{-\infty}^{\infty} \tilde{\lambda}_l^{*(n)}(k_x) \tilde{\lambda}_L^{(n)}(k_x) \right. \\ &\quad \cdot \tilde{\lambda}_m^{*(n)}(k_y) \tilde{\lambda}_M^{(n)}(k_y) \\ &\quad \cdot \left\{ \int dz \int dz' \tilde{G}(k_x, k_y; z, z') \lambda_k^{(n)}(z/\delta z) \right. \\ &\quad \left. \cdot \lambda_K^{(n)}(z'/\delta z) \right\} dk_x dk_y \right] \cdot \sigma_{LMK} E_{LMK}. \end{aligned} \quad (54)$$

The vector-matrix equation becomes, therefore:

$$E_{lmk}^{(i)} = E_{lmk} + \sum_L \sum_M \sum_K G_{kk}(l-L, m-M) \cdot \sigma_{LMK} E_{LMK}, \quad (55)$$

where,

$$\begin{aligned} E_{lmk}^{(i)} &= \frac{1}{\delta x \delta y \delta z} \int \int \int E^{(i)}(r) \lambda_l^{(n)}(x/\delta x) \\ &\quad \cdot \lambda_m^{(n)}(y/\delta y) \lambda_k^{(n)}(z/\delta z) dr \\ \sigma_{LMK} &= \frac{1}{\delta x \delta y \delta z} \int \int \int \sigma(r) [F_L^{(n)}(x/\delta x)]^2 \\ &\quad \cdot [F_M^{(n)}(y/\delta y)]^2 [F_K^{(n)}(z/\delta z)]^2 dr \\ G_{kk}(l-L, m-M) &= \int \int_{-\infty}^{\infty} \tilde{\lambda}_l^{*(n)}(k_x) \tilde{\lambda}_L^{(n)}(k_x) \tilde{\lambda}_m^{*(n)}(k_y) \\ &\quad \cdot \tilde{\lambda}_M^{(n)}(k_y) \tilde{\Gamma}_{kk}(k_x, k_y) dk_x dk_y \\ \tilde{\Gamma}_{kk}(k_x, k_y) &= \int dz \int dz' \tilde{G}(k_x, k_y; z, z') \\ &\quad \cdot \lambda_k^{(n)}(z/\delta z) \lambda_k^{(n)}(z'/\delta z). \end{aligned} \quad (56)$$

Note that

$$\tilde{\lambda}_l^{*(n)}(k_x) \tilde{\lambda}_L^{(n)}(k_x) = e^{jk_x(L-l)} |\tilde{\lambda}_n(k_x)|^2,$$

which is a function of the difference of the two arguments, l and L . If $n = 0$ we get the expansion and testing in pulse-functions that were used in [5] and [6].

The computation of $\tilde{\Gamma}_{kk}(k_x, k_y)$ in (47) and (56) has already been performed when $n = 0$, i.e., for pulse functions [5], [6]. Its computation for higher-order splines will be discussed in the next section.

V. CALCULATION OF $\tilde{\Gamma}_{kk}(k_x, k_y)$

The expressions for $\tilde{\Gamma}_{kk}(k_x, k_y)$ are given in (47) and (56). We will evaluate these expressions for first-order splines and their reciprocal-bases, but the method of calculation is quite general and can be easily applied to higher-order splines and reciprocal-bases.

It follows from the discussion of the Green's dyad in (3)-(5) that the relevant parts of $\tilde{G}(z, z')$ are given by the four expressions

- 1) $e^{-\alpha|z-z'|}$
- 2) $\pm e^{-\alpha|z-z'|} (+) : z > z', (-) : z < z'$
- 3) $\delta(z - z')$
- 4) $e^{-\alpha(z+z')}$.

We will compute the integrals in (47), first, and then (56). The integral of the first function in the above list is given by

$$F_1 = \int dz' \int dz F_k(z/\delta z) F_K(z'/\delta z) e^{-\alpha|z-z'|}, \quad (57)$$

where we have dropped the superscript n , because the order of the spline is one. F_1 is symmetric in k, K .

The computation of F_1 starts with

$$f_K(z) = \int dz' e^{-\alpha|z-z'|} F_K(z'/\delta z). \quad (58)$$

This is a convolution integral, and can be easily evaluated using Fourier transforms. We have the Fourier transform relation

$$e^{-\alpha|z-z'|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + \xi^2} e^{-j\xi(z-z')} d\xi, \quad (59)$$

which can be easily verified by simple contour integration in the complex-plane, or by looking it up in a table of integrals.

Thus, by the convolution-transform theorem, we have

$$\begin{aligned} f_K(z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + \xi^2} \tilde{\pi}_2(z'/\delta z - K) e^{-j\xi z} d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + \xi^2} \delta z e^{j(K+1)\delta z \xi} \left(\frac{\sin(\xi \delta z/2)}{\xi \delta z/2} \right)^2 \\ &\quad \cdot e^{-j\xi z} d\xi, \end{aligned} \quad (60)$$

where we have used (8), with a change of notation.

Now, $F_1 = \int F_k(z/\delta z) f_K(z) dz$. When (60) is substituted into this expression, and the order of integration with respect to z and ξ is interchanged, we find the interior in-

tegral to be $\int F_k(z/\delta z) \exp(-j\xi z) dz = \hat{\pi}_2^*(z/\delta z - k)$, which is the complex-conjugate of the transform in (8). Therefore, when this is substituted into the expression for F_1 , we get

$$\begin{aligned} F_1 &= \frac{\delta z^2}{2\pi} \int_{-\infty}^{\infty} e^{-j\delta z(k-K)\xi} \frac{2\alpha}{\alpha^2 + \xi^2} \left(\frac{\sin(\xi\delta z/2)}{\xi\delta z/2} \right)^4 d\xi \\ &= \frac{\delta z^2}{2\pi} \int_{-\infty}^{\infty} e^{-jh(k-K)} \frac{2\beta}{\beta^2 + h^2} \\ &\quad \cdot \left(\frac{e^{j2h} - 4e^{jh} + 6 - 4e^{-jh} + e^{-j2h}}{h^4} \right) dh \\ &= G_1 + G_2 + G_3 + G_4 + G_5. \end{aligned} \quad (61)$$

We have defined $h = \delta z\xi$, $\beta = \alpha\delta z$, and

$$\begin{aligned} G_1 &= \frac{\delta z^2}{2\pi} \int_{-\infty}^{\infty} \frac{2\beta}{\beta^2 + h^2} \frac{e^{-jh(k-K-2)}}{h^4} dh \\ G_2 &= \frac{-4\delta z^2}{2\pi} \int_{-\infty}^{\infty} \frac{2\beta}{\beta^2 + h^2} \frac{e^{-jh(k-K-1)}}{h^4} dh \\ G_3 &= \frac{6\delta z^2}{2\pi} \int_{-\infty}^{\infty} \frac{2\beta}{\beta^2 + h^2} \frac{e^{-jh(k-K)}}{h^4} dh \\ G_4 &= \frac{-4\delta z^2}{2\pi} \int_{-\infty}^{\infty} \frac{2\beta}{\beta^2 + h^2} \frac{e^{-jh(k-K+1)}}{h^4} dh \\ G_5 &= \frac{\delta z^2}{2\pi} \int_{-\infty}^{\infty} \frac{2\beta}{\beta^2 + h^2} \frac{e^{-jh(k-K+2)}}{h^4} dh. \end{aligned} \quad (62)$$

Each of the integrands in (62) has a simple pole at $\pm j\beta$, and a fourth-order pole at the origin. (Note, however, that the integrand of (61) is analytic at the origin.) The pole at the origin is avoided by a small semi-circle in the lower half-plane. Because (61) is analytic at the origin, we need not include an integral over this small semi-circle, since it vanishes when all the G integrals are added together. For higher-order splines the singularity at the origin would be of higher-order, also.

We show the complex-plane contours and poles in Fig. 3 for calculating G_1 . This is a prototype for the other G 's.

We apply the Cauchy integral formula for the computation of the G 's:

$$\begin{aligned} \oint \frac{f(z)}{z - z_0} dz &= \pm j2\pi f(z_0) \\ \oint \frac{f(z)}{(z - z_0)^4} dz &= \pm j2\pi \frac{1}{6} f'''(z_0), \end{aligned} \quad (63)$$

where (+) goes with integration in the positive (CCW) direction (which is in the upper half-plane of Fig. 3), and

$$\begin{aligned} F_1 &= \delta z^2 \left[\frac{2e^{-2\beta} - 8e^{-\beta} + 6}{\beta^4} - \frac{4}{\beta^3} + \frac{4}{3\beta} \right]; \quad k - K = 0 \\ &= \delta z^2 \left[\frac{e^{-3\beta} - 4e^{-2\beta} + 7e^{-\beta} - 4 + 2\beta + 1/3\beta^3}{\beta^4} \right]; \quad k - K = 1 \\ &= \delta z^2 \left[\frac{e^{-\beta(k-K-2)} - 4e^{-\beta(k-K-1)} + 6e^{-\beta(k-K)} - 4e^{-\beta(k-K+1)} + e^{-\beta(k-K+2)}}{\beta^4} \right]; \quad k - K \geq 2. \end{aligned} \quad (65)$$

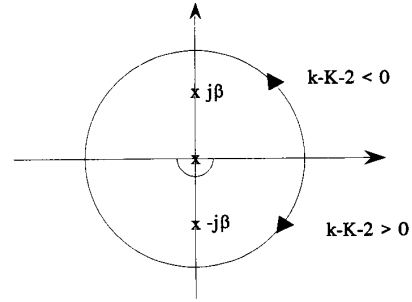


Fig. 3. Showing the location of the poles, and the contours for computing G_1 . The contours for the other G 's are similar.

(-) with integration in the negative direction (CW) (lower half-plane).

The results are

$$\begin{aligned} G_1^{(U)} &= \delta z^2 \left[\frac{e^{\beta(k-K-2)}}{\beta^4} - \frac{2(k-K-2)}{\beta^3} - \frac{(k-K-2)^3}{3\beta} \right]; \quad k - K - 2 < 0 \\ G_1^{(L)} &= \delta z^2 \frac{e^{-\beta(k-K-2)}}{\beta^4}; \quad k - K - 2 > 0 \\ G^{(U)} &= -4\delta z^2 \left[\frac{e^{\beta(k-K-1)}}{\beta^4} - \frac{2(k-K-1)}{\beta^3} - \frac{(k-K-1)^3}{3\beta} \right]; \quad k - K - 1 < 0 \\ G_2^{(L)} &= -4\delta z^2 \frac{e^{-\beta(k-K-1)}}{\beta^4}; \quad k - K - 1 > 0 \\ G_3^{(L)} &= 6\delta z^2 \frac{e^{-\beta(k-K)}}{\beta^4}; \quad k - K > 0 \\ G_4^{(L)} &= -4\delta z^2 \frac{e^{-\beta(k-K+1)}}{\beta^4}; \quad k - K + 1 > 0 \\ G_5^{(L)} &= \delta z^2 \frac{e^{-\beta(k-K+2)}}{\beta^4}; \quad k - K + 2 > 0, \end{aligned} \quad (64)$$

where the superscript, (U), denotes that the contour is closed in the upper half-plane, and (L) denotes closure in the lower half-plane.

In Fig. 4 we show the $k - K$ axis, together with the G 's that contribute to F_1 within each interval. Of course, we need show only the positive $k - K$ axis, because we have already shown that F_1 is an even function of $k - K$.

When we sum the appropriate contributions, we get

This completes the calculation of F_1 . Having an explicit form for F_1 allows us to compute F_2 easily. We start with the observation

$$\begin{aligned} \pm e^{-\alpha|z-z'|} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{j2\xi}{\alpha^2 + \xi^2} e^{-j\xi(z-z')} d\xi \\ &= \frac{-1}{\alpha} \frac{d}{d(z-z')} e^{-\alpha|z-z'|}. \end{aligned} \quad (66)$$

This eventually translates into the result $F_{2kk} = -(1/\beta)dF_1/d(k-K)$. This shows that $F_{2kk} = -F_{2Kk}$; i.e., the matrix is antisymmetric. We make use of these results by simply differentiating the G 's of (64), and then dividing by $-\beta$. When these results are put into Fig. 4, we get

$$\begin{aligned} F_2 &= 0; \quad k-K=0 \\ &= \delta z^2 \left[\frac{e^{-3\beta} - 4e^{-2\beta} + 5e^{-\beta} - 2 + \beta^2}{\beta^4} \right]; \quad k-K=1 \\ &= \delta z^2 \left[\frac{e^{-\beta(k-K-2)} - 4e^{-\beta(k-K-1)} + 6e^{-\beta(k-K)} - 4e^{-\beta(k-K+1)} + e^{-\beta(k-K+2)}}{\beta^4} \right]; \quad k-K \geq 2. \end{aligned} \quad (67)$$

These are the two most complicated results; the rest of the calculations proceed much more simply.

$$\begin{aligned} F_3 &= \int dz \int dz' \delta(z-z') F_k(z/\delta z) F_K(z'/\delta z) \\ &= \int dz F_k(z/\delta z) F_K(z/\delta z). \end{aligned} \quad (68)$$

Upon taking into account the triangular shape of the first-order spline, $F_k(z/\delta z)$, together with its overlap with

$F_K(z/\delta z)$, it is easy to compute the following result

$$\begin{aligned} F_3 &= \frac{\delta z}{6}, \quad |k-K|=1 \\ &= 4 \frac{\delta z}{6}, \quad |k-K|=0 \\ &= 0, \quad \text{otherwise.} \end{aligned} \quad (69)$$

Function F_4 is simply the product of two single integrals:

$$F_4 = \int dz e^{-\alpha z} F_k(z/\delta z) \int dz' e^{-\alpha z'} F_K(z'/\delta z).$$

A straight-forward calculation gives

$$\int dz e^{-\alpha z} F_k(z/\delta z) = \delta z e^{-\alpha \delta z k} \left[\frac{1 - e^{-\alpha \delta z}}{\alpha \delta z} \right]^2,$$

from which we conclude that

$$F_4 = \delta z^2 \left[\frac{1 - e^{-\alpha \delta z}}{\alpha \delta z} \right]^4 e^{-\alpha \delta z(k+K)}. \quad (70)$$

The integrals that appear in (56), and involve the first-order reciprocal-basis, are computed in the same manner as above. Of course, where appropriate we use the Fourier transforms of (32) and (33), instead of (8). The results, using the same notation as before, are

$$\begin{aligned} F_1 &= \delta z^2 \left[\frac{72 - 20\beta^2 + 8\beta^3}{\beta^4} - \frac{16\beta^2 + 72\beta + 72}{\beta^4} e^{-\beta} \right]; \quad k-K=0 \\ &= \delta z^2 \left[\frac{72 - 20\beta^2}{\beta^4} e^{-\beta} - \frac{8\beta^2 - 36\beta + 36}{\beta^4} - \frac{8\beta^2 + 36\beta + 36}{\beta^4} e^{-2\beta} \right]; \quad k-K=1 \\ &= \delta z^2 \left[\frac{72 - 20\beta^2}{\beta^4} e^{-\beta(k-K)} - \frac{8\beta^2 + 36\beta + 36}{\beta^4} e^{-\beta(k-K+1)} \right. \\ &\quad \left. - \frac{8\beta^2 - 36\beta + 36}{\beta^4} e^{-\beta(k-K-1)} \right]; \quad k-K \geq 2 \\ F_2 &= 0; \quad k-K=0 \\ &= \delta z^2 \left[\frac{72 - 20\beta^2}{\beta^4} e^{-\beta} - \frac{8\beta^2 - 36\beta + 36}{\beta^4} - \frac{8\beta^2 + 36\beta + 36}{\beta^4} e^{-2\beta} \right]; \quad k-K=1 \\ &= \delta z^2 \left[\frac{72 - 20\beta^2}{\beta^4} e^{-\beta(k-K)} - \frac{8\beta^2 + 36\beta + 36}{\beta^4} e^{-\beta(k-K+1)} \right. \\ &\quad \left. - \frac{8\beta^2 - 36\beta + 36}{\beta^4} e^{-\beta(k-K-1)} \right]; \quad k-K \geq 2 \\ F_3 &= 4\delta z \delta_{kK} \\ F_4 &= \left[-2 - 4e^{-\alpha \delta z} + 6 \left(\frac{1 - e^{-\alpha \delta z}}{\alpha \delta z} \right)^2 \right] \frac{e^{-\alpha \delta z(k+K)}}{\alpha^2}. \end{aligned} \quad (71)$$

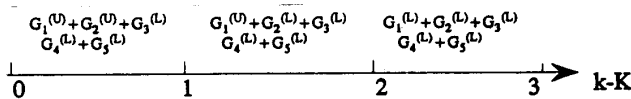


Fig. 4. Showing the $k - K$ axis, together with the G 's that contribute to F_i within each interval.

VI. COMMENTS AND CONCLUSIONS

We have applied this theory to the case in which the expansion and testing functions are both pulses (the pulse functions are their own reciprocal bases) [5], [6]. Of course, this gives us the familiar Galerkin variant of the method of moments, and our results have been quite good. By using higher-order splines and their reciprocal bases in the manner suggested in this paper, we retain the same structure for the resulting integral equation as in the pulse-pulse situation, but, of course, the field solutions will be smoother.

There are several desirable features of the discretized integral equations that result from the approach of this paper. The first term on the right-hand side of (46) and (55) is always a diagonal matrix, regardless of the order of the expansion function. If we used a Galerkin approach, this term would be a multi-diagonal matrix, depending upon the order of the splines used for expansion and testing. For example, the matrix would be tri-diagonal for first-order splines, penta-diagonal for second-order splines, etc.

A second feature is that we have a well-defined, optimal way of defining the discretized coefficient, σ_{lmk} , that appears in the integral equations. We do not need to use a "stair-step" approximation to the discretized region, or rely on an inaccurate *ad hoc* method for defining the discretized material parameters. We have presented a consistent way of defining these parameters when used in conjunction with higher-order splines.

Similarly, the computation of the matrix elements, $G_{kk}(l - L, m - M)$, is made quite systematic. One can use the same piece of code written for the pulse-pulse combination, and substitute the necessary Fourier transforms for the higher-order splines.

We have used the method of higher-order splines and their reciprocal bases in discretizing transient boundary-integral equations, as well [9], [10]. In these papers we used one-dimensional second-order splines and their reciprocal bases to discretize the time-domain response of the space-time boundary-integral equation. The method produced stable and accurate results. We are preparing a paper to give more details about this application.

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