

A Low-Rank IE-QR Algorithm for Matrix Compression in Volume Integral Equations

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Abstract—A single-level matrix compression algorithm based on pivoted QR factorization with partial matrix assembling, which exploits the rank deficiency of matrix blocks for physically separated groups of basis functions, is presented for the volume integral equation solution of electromagnetic scattering from arbitrarily shaped dielectric bodies. For a system of N equations, an amount of work of the order $O(N^2)$ has traditionally been required by the method of moments (MoM). The algorithm of the present paper reduces both computational complexity and storage requirement to $O(N^{1.5})$ with relatively less dependence on the integral equation kernel. Hence, the proposed algorithm is more practical for large-scale problems and can be implemented in a wide range of applications with few or no modifications.

Index Terms—Electromagnetic scattering, integral equations, method of moments, QR factorization.

I. INTRODUCTION

ELECTROMAGNETIC scattering from arbitrarily shaped inhomogeneous dielectric bodies has been an interesting topic due to its importance in problems that include propagation through inhomogeneities. These types of problems can be formulated in the form of a volume integral equation (VIE) and solved by the method of moments (MoM). However, solving large-scale electromagnetic problems by MoM suffers from $O(N^2)$ computational complexity and storage requirements, where N is the number of unknowns.

There have been a number of successful algorithms that reduce the computational complexity of the integral equation methods, the most prominent of these being the fast multipole method (FMM) [1]. For electromagnetic volume scattering, multilevel FMM exhibits $O(N)$ computational complexity [2]. However, the algorithm highly depends on the integral equation kernel making a single scheme that is difficult to generalize for a wide class of applications. For example, implementation of the FMM for the multilayered medium Green's function is much more involved than that for the free-space Green's function. Another approach is based on representing the matrix blocks of the MoM matrix through low-rank approximations [3]–[5]. The matrix compression algorithms presented in [3] and [5] require the off-diagonal matrix blocks to be fully assembled prior to compression. Hence, they have $O(N^2)$ computational complexity. The integral equation solver “IES³,” which is given in [4], claims to reduce the computational complexity to $O(N \log N)$ through hierarchical singular value

decomposition (SVD) with a reduced sampling procedure. However, the reduced sampling procedure employed in [4] is not fully described.

The low-rank IE-QR algorithm presented in this paper is also based on low-rank matrix representations but with a new and efficient pivoted QR factorization approach that requires the matrix blocks that are only partially formed. The block IE-QR approach seeks to reduce the linear dependencies (redundant information) in blocks of the MoM matrix corresponding to physically separated regions of basis functions. Our approach uses a dual-rank modified Gram–Schmidt (MGS) algorithm that selects and generates only r rows and columns of the block matrix, where r is the approximate rank of the original block matrix. The resulting QR factorization may then be used to compute the matrix-vector product operation with far less complexity than $O(N^2)$. Used in an iterative solver with an appropriate preconditioner, this leads to a highly efficient integral equation method for general electromagnetic radiation and scattering problems. Through numerical examples, the computational complexity has been shown to be close to $O(N^{1.5})$.

II. VOLUME INTEGRAL EQUATION FORMULATION

Let V denote the volume of a scatterer characterized by its relative permittivity $\varepsilon(\vec{r})$ and relative permeability $\mu(\vec{r})$, which are continuous inside V , and let the background medium be free space. Applying the volume equivalence principle and the continuity equation, the scatterer can be replaced with equivalent electric current and charge densities [6]

$$\vec{J}(\vec{r}) = j\omega\kappa(\vec{r})\vec{D}(\vec{r}) \quad (1)$$

$$\rho(\vec{r}) = -\nabla\kappa(\vec{r}) \cdot \vec{D}(\vec{r}) - \kappa(\vec{r})\nabla \cdot \vec{D}(\vec{r}) \quad (2)$$

where $\kappa(\vec{r}) = (\varepsilon(\vec{r}) - \varepsilon_o)/(\varepsilon(\vec{r}))$ is the contrast ratio, $\vec{D}(\vec{r})$ is the total electric flux density, and ω is the frequency of operation. Decomposing the total field into incident and scattered fields, we have

$$\vec{E}(\vec{r}) = \vec{E}^i(\vec{r}) + \vec{E}^s(\vec{r}) \quad (3)$$

where $\vec{E}^i(\vec{r})$, $\vec{E}^s(\vec{r})$, and $\vec{E}(\vec{r})$ are the incident, scattered, and total fields at position $\vec{r}$, respectively. The scattered field can be expressed by mixed potential formulation [6]

$$\vec{E}^s = -j\omega\vec{A}(\vec{r}) - \nabla\phi(\vec{r}) \quad (4)$$

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where

$$\begin{aligned}\vec{A}(\vec{r}) &= \frac{\mu_o}{4\pi} \int_V \vec{J}(\vec{r}') \frac{e^{-jk_o|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} dv' \\ \phi(\vec{r}) &= \frac{\mu_o}{4\pi} \int_V \rho(\vec{r}') \frac{e^{-jk_o|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} dv'.\end{aligned}\quad (5)$$

Discretization of the volume by tetrahedral elements and interpolation of $\vec{D}(\vec{r})$ by the first-order divergence conforming vector basis functions followed by testing of (3) with the basis function chosen leads to a system of linear equations that can be written in matrix form as

$$\mathbf{Z} \cdot \mathbf{D} = \mathbf{Y} \quad (6)$$

where N is the total number of unknowns, $\mathbf{Z}$ is the $N \times N$ impedance matrix, and $\mathbf{D}$ and $\mathbf{Y}$ are vectors of length N [7].

III. LOW-RANK IE-QR ALGORITHM

The low-rank IE-QR algorithm is based on the rank deficiency of matrix blocks for physically separated groups of basis functions. When the separation distance between two groups of basis functions is greater than their physical sizes, the effective information between these two groups is limited, in other words, the number of independent vectors to include the majority of couplings between these two groups, the rank, in general is much smaller than the number of basis functions in the transmitting (source) or the receiving (observation) group. Hence, the matrix block $\mathbf{Z}_{ij}^{m \times n}$ that represents the interaction between a receiving group i and a transmitting group j , ($i, j = 1, 2, \dots, N_g; i \neq j$) can be approximated by the product of $\mathbf{Q}_{ij}^{m \times r}$ and $\mathbf{R}_{ij}^{r \times n}$ matrices, where m and n are the number of basis functions in the transmitting and receiving groups, respectively, r is the approximate rank of $\mathbf{Z}_{ij}^{m \times n}$, and N_g is the number of groups, namely

$$\mathbf{Z}_{ij}^{m \times n} = [\mathbf{z}_1 \quad \mathbf{z}_2 \quad \dots \quad \mathbf{z}_n] = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{bmatrix} \approx \mathbf{Q}_{ij}^{m \times r} \cdot \mathbf{R}_{ij}^{r \times n}. \quad (7)$$

Here, $\mathbf{z}_p$ is the column vector due to the p th basis function in the transmitting group (transmitter), and ν_q is the row vector due to the q th basis function in the receiving group (receiver). In a brute force QR factorization approach, each full block must be computed first so that MGS algorithm can be applied [8]. This leads to $O(N^2)$ complexity in the compression stage. Our approach uses a dual-rank MGS algorithm that selects and generates r rows and columns of the block matrix with the least linear dependence. Hence, our QR factorization has $O(r(m+n))$ complexity, compared with $O(mn)$ complexity for a brute force approach, where r is generally much less than m and n for well-separated groups. Likewise, the storage requirement is reduced from mn to $(m+n)r$. The numerical results show that the overall computational complexity and storage requirement tend to stay close to $O(N^{1.5})$.

The low-rank IE-QR algorithm consists of four steps: column index selection, updating the $\mathbf{Q}$ matrix, row index selection, and

TABLE I
LOW-RANK IE-QR ALGORITHM WITH MATRIX BLOCKS PARTIALLY FORMED

Column Index Selection	
Let $\mathbf{B}^{(k)} = \Pi_k \mathbf{Z}(1:k; 1:n)$, we compute $p = \text{COL}_{k-1}(\mathbf{B}^{(k)})$. Then the permutation matrix Π_k^c is an $n \times n$ identity with p and k columns interchanges.	
Updating Q	
Update $\Pi_k \mathbf{Z}$ to $\Pi_{k+1/2} \mathbf{Z} = (\Pi_k \mathbf{Z}) \Pi_k^c =$	$\begin{bmatrix} X & \dots & \dots & \dots & \dots & X \\ \vdots & & & & & \vdots \\ X & \dots & \dots & \dots & \dots & X \\ X & \dots & X & O & \dots & O \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X & \dots & X & O & \dots & O \end{bmatrix}_k$. Let
$q_k = \Pi_{k+1/2} \mathbf{Z}(1:m; k)$, then compute $q_k^\perp = (\mathbf{Q}^{(k-1)})^\perp q_k$, $\delta = \frac{\ q_k^\perp\ }{\ q_k\ }$, and	
$q_k = \frac{q_k^\perp}{\ q_k^\perp\ }$. Update from $\mathbf{Q}^{(k-1)}$ to $\mathbf{Q}^{(k)}$.	
Row Index Selection	
Compute $\bar{p} = \text{ROW}_{k+1}(\mathbf{Q}^{(k)})$, and the row permutation matrix Π_{k+1}^r is an $m \times m$ identity with $\bar{p}$ and $k+1$ rows interchanged. Update $\Pi_{k+1/2} \mathbf{Z}$ to $\Pi_{k+1} \mathbf{Z} = \Pi_{k+1}^r (\Pi_{k+1/2} \mathbf{Z})$.	
Convergence Check and Computing R	
If $\delta < \tau$, where τ is a prescribed tolerance, then compute $\mathbf{R}^{(k)}$ according to (14). Find J indexes, $\{i_1, i_2, \dots, i_J\} = \text{ROW}_{k,J}(\mathbf{Q}^{(k)})$, and compute $\eta = \frac{\ \mathbf{E}\ _{\infty, \{i_1, i_2, \dots, i_J\}}}{\ \Pi_{k+1} \mathbf{Z}\ _{\infty}}$. If $\eta \leq \tau$, then we are finished. Otherwise, continue IE-QR	

computing the $\mathbf{R}$ matrix. At the k th step of the algorithm, the index of the column vector in $\mathbf{Z}$ that is the most linearly independent on the previously selected $k-1$ columns is determined through the MGS procedure. $\mathbf{Q}$ is updated with this column vector by the MGS algorithm. The index of the most significant row vector is then the index of the row vector with the largest norm in $\mathbf{Q}$. This iterative scheme continues until $\mathbf{Q}$ matrix is approximated within prescribed tolerance. As the last step, the $\mathbf{R}$ matrix is constructed by solving the r equations extracted from (7).

Below, we give some mathematical definitions to use in the algorithmic prescription of low rank IE-QR algorithm given in Table I.

Definition 1 (Modified Gram-Schmidt): Given n linearly independent column vectors $\nu_1, \dots, \nu_n$, we denote the orthonormal matrix $\mathbf{Q} = \text{MGS}\{\nu_1, \dots, \nu_n\}$ that is obtained by these n vectors through the modified Gram-Schmidt process, namely

$$\begin{aligned}\mathbf{Q} &= [\mathbf{q}_1 \quad \dots \quad \mathbf{q}_n] \\ \mathbf{q}_i &= \mathbf{v}_i, \quad \mathbf{q}_i = \mathbf{q}_i - (\mathbf{q}_j^t \mathbf{q}_i) \mathbf{q}_j; j = 1 \dots (i-1), \\ \mathbf{q}_i &= \frac{\mathbf{q}_i}{\|\mathbf{q}_i\|}.\end{aligned}\quad (8)$$

Definition 2 (Orthogonal Component): Given an orthonormal matrix $\mathbf{Q}^{m \times n} = [\mathbf{q}_1 \cdots \mathbf{q}_n]$ and a column vector $\mathbf{v}$, we define $\mathbf{Q}^\perp \mathbf{v}$ by

$$\mathbf{Q}^\perp \mathbf{v} = \mathbf{v} - \mathbf{Q}(\mathbf{Q}^T \mathbf{v}). \quad (9)$$

It is clear that $\mathbf{Q}^\perp \mathbf{v} \perp \mathbf{q}_i; i = 1 \cdots n$.

Definition 3: We say that a matrix $\tilde{\mathbf{Z}}^{m \times n}$ is an $i \times j$ approximate of $\mathbf{Z}^{m \times n}$ if

$$\begin{aligned} \tilde{\mathbf{Z}}(1:m; 1:j) &= \mathbf{Z}(1:m; 1:j); \tilde{\mathbf{Z}}(1:m; 1:j) \\ &= \mathbf{Z}(1:m; 1:j) \end{aligned} \quad (10)$$

and we write $\tilde{\mathbf{Z}} = \Theta_{i \times j}(\mathbf{Z})$.

Definition 4 (Column Index Selection): Given a $m \times n$ rectangular matrix, $\mathbf{B}^{m \times n} = [\mathbf{b}_1 \cdots \mathbf{b}_n]$ and assuming $\mathbf{Q} = \text{MGS}\{\mathbf{b}_1 \cdots \mathbf{b}_k\}$ ($k < m, n$) exists, we define $\text{COL}_k(\mathbf{B})$ as the minimum index $p, k < p \leq n$ such that

$$\|\mathbf{Q}^\perp \mathbf{b}_p\| = \max \{ \|\mathbf{Q}^\perp \mathbf{b}_{k+1}\|, \|\mathbf{Q}^\perp \mathbf{b}_n\| \}. \quad (11)$$

Definition 5 (Row Index Selection): Given an $m \times n$ rectangular matrix and its row partition $\mathbf{B}^{m \times n} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_m \end{bmatrix}$, we define

$\{i_1, i_2, \dots, i_p\} = \text{ROW}_{k,p}(\mathbf{B})$ according to $k < i_1, i_2, \dots, i_{m-k} \leq m$

$$\|\mathbf{b}_{i_1}\| \geq \|\mathbf{b}_{i_2}\| \geq \dots \geq \|\mathbf{b}_{i_p}\| \geq \dots \geq \|\mathbf{b}_{i_{m-k}}\|. \quad (12)$$

Definition 6 (Approximate ∞ Norm): Given an $m \times n$ rectangular matrix $\mathbf{B}^{m \times n}$ and a series of indexes $i_1, i_2, \dots, i_p$, we define an approximate ∞ norm of $\mathbf{B}^{m \times n}$ by

$$\|\mathbf{B}\|_{\infty, \{i_1, i_2, \dots, i_p\}} = \max_{i=i_1, i_2, \dots, i_p} \sum_{j=1}^n |\mathbf{B}_{ij}| \quad (13)$$

The following is a straightforward theorem; therefore, we simply state it.

Theorem 1: Given a matrix $\mathbf{Z}^{m \times n}$ and its column partition $\mathbf{Z} = [\mathbf{z}_1 \mathbf{z}_2 \cdots \mathbf{z}_n]$, the matrix product $\tilde{\mathbf{Z}} = \mathbf{Q}^{m \times k} \mathbf{R}^{k \times n}$ with (14), shown at the bottom of the page, is a $k \times k$ approximate of $\mathbf{Z}^{m \times n}$, assuming $\mathbf{Q}(1:k; 1:k)$ is nonsingular. That is, $\mathbf{Q}^{m \times k} \mathbf{R}^{k \times n} = \Theta_{k \times k}(\mathbf{Z}^{m \times n})$. Note that the error for the $\mathbf{Q}^{m \times k} \mathbf{R}^{k \times n}$ approximate is

$$\begin{aligned} \mathbf{E}^{m \times n} &= \mathbf{Z}^{m \times n} - \mathbf{Q}^{m \times k} \mathbf{R}^{k \times n} \\ &= \begin{bmatrix} \underbrace{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}_k & \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \end{bmatrix} \cdot \mathbf{E}^{(m-k) \times (n-k)} \quad (15) \\ &\quad \left(= \mathbf{Z}(k+1:m; k+1:n) \right) \\ &\quad \left(-\mathbf{Q}(k+1:m; 1:k) \mathbf{R}_{12} \right) \end{bmatrix} \end{aligned}$$

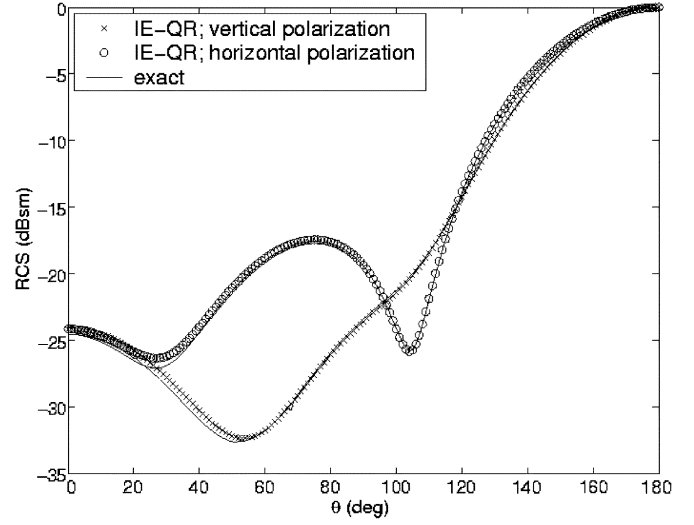


Fig. 1. Bi-static RCS pattern for vertical and horizontal polarization.

Table I gives a summary of the IE-QR algorithm with the matrix blocks partially formed. At the k^{th} step, we have explicitly formed an $k \times (k-1)$ approximate of the permuted matrix, namely

$$\begin{aligned} \Pi_k \mathbf{Z} &= (\Pi_k^r \cdots \Pi_1^r) \mathbf{Z} (\Pi_1^c \cdots \Pi_{k-1}^c) \\ &= \begin{bmatrix} \underbrace{\begin{bmatrix} X & \cdots & X \\ \vdots & \ddots & \vdots \\ X & \cdots & X \end{bmatrix}}_{k-1} & \begin{bmatrix} O & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & O \end{bmatrix} \end{bmatrix} \quad (16) \end{aligned}$$

In (16), the entries marked by X are available, i.e. have been computed, whereas the O entries are not. Moreover, an orthonormal matrix $\mathbf{Q}^{(k-1)}$, with dimensions $m \times (k-1)$, has been constructed through previous $(k-1)$ steps.

IV. NUMERICAL RESULTS

To prove the accuracy of low-rank approximation of the single-level IE-QR algorithm, we studied a dielectric sphere example. The sphere has a radius of 10 cm and a dielectric constant of $\epsilon_r = 1.63$ and is illuminated at $\theta = 0^\circ$. A recursive partitioning algorithm has been implemented to group the basis functions in the problem geometry. Generalized minimum residual (GMRES) with tolerance 0.01 has been used as an iterative solver. The tolerance to stop the IE-QR process is also chosen 0.01. At $f = 1500$ MHz, which corresponds to $d = \lambda_0$,

$$\begin{aligned} \mathbf{Q}^{m \times k} &= \text{MGS}\{\mathbf{z}_1 \cdots \mathbf{z}_k\} \\ \mathbf{R}^{k \times n} &= (\mathbf{Q}(1:k; 1:k))^{-1} \mathbf{Z}(1:k; 1:n) \\ &= \begin{bmatrix} \mathbf{R}_{11}^{k \times k} & \mathbf{R}_{12}^{k \times (n-k)} \\ \left((\mathbf{Q}(1:k; 1:k))^{-1} \mathbf{Z}(1:k; 1:k) \right) & \left((\mathbf{Q}(1:k; 1:k))^{-1} \mathbf{Z}(1:k; k+1:n) \right) \end{bmatrix} \quad (14) \end{aligned}$$

TABLE II
PERFORMANCE OF SINGLE-LEVEL IE-QR ALGORITHM

N	Memory (MB)	Assembly Time	Solver Time	Iteration
6262	134.340	00:11:21	00:00:05	5
10462	274.664	00:23:09	00:00:17	10
14612	443.612	00:37:57	00:00:31	11
18436	636.624	00:56:06	00:00:37	9
24268	943.096	01:25:02	00:00:59	10
29026	1248.380	01:52:05	00:01:33	12

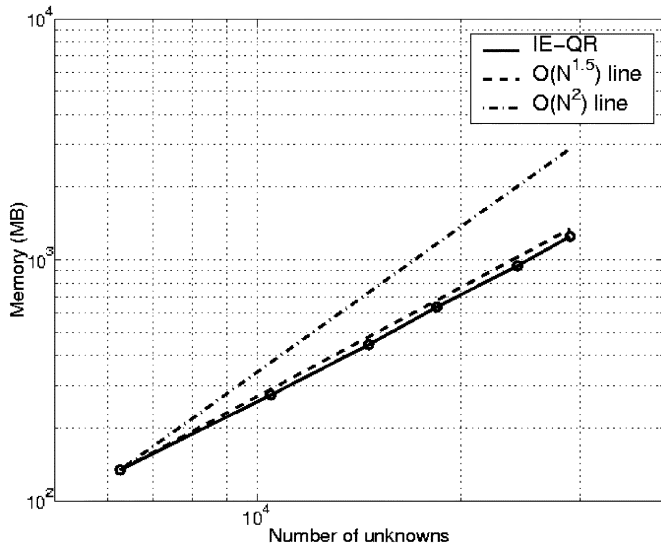


Fig. 2. Storage requirement for the low-rank IE-QR algorithm.

where d is the diameter, the sphere is discretized into 6794 tetrahedrons and 14 612 unknowns. The number of groups for this problem is 128, with the minimum and maximum group sizes being 58 and 179, respectively. We observe that for physically separated groups, approximate numerical rank ranges from 6 to 20. This observation demonstrates that block matrices for the physically separated groups of basis functions are indeed rank-deficient. In Fig. 1, bi-static RCS patterns obtained from the numerical and exact solutions are compared. Agreement between the two solutions proves the accuracy of low-rank approximation. We should note that for this example, the compression rate is 26%.

To demonstrate the numerical efficiency of the current algorithm, we implemented the algorithm on different meshes with increasing number of unknowns while keeping the average mesh size almost constant. Table II summarizes the computational details of this study. In Figs. 2 and 3, the storage requirement and computational complexity of the matrix compression are plotted, respectively. Comparison with the reference lines shows that complexity tends to stay close to $O(N^{1.5})$.

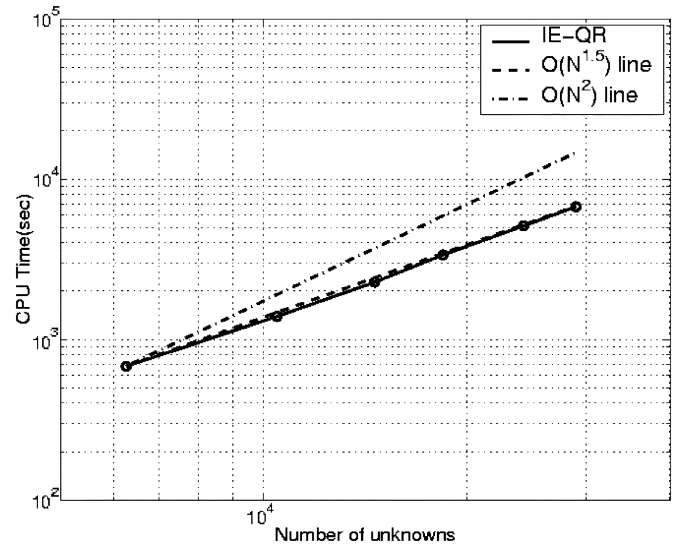


Fig. 3. CPU Time for matrix compression in low-rank IE-QR algorithm.

V. CONCLUSION

A single-level low-rank IE-QR algorithm has been presented to speed up the volume integral equation solution and to reduce the memory requirement for the electromagnetic scattering from dielectric bodies. The important advantages of this algorithm are that it neither depends on the exact knowledge of the Green's functions like the FMM nor requires a full assembly of the impedance matrix. Hence, the computational complexity of the matrix compression stage is reduced. The numerical results show that computational complexity stays close to $O(N^{1.5})$. The authors are currently working on the multilevel IE-QR algorithm to make the current method compatible with other fast integral equation methods for general electromagnetic radiation/scattering problems.

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