

# Splines and Their Reciprocal-Bases in Volume-Integral Equations

H. A. Sabbagh

Sabbagh Associates, Inc.  
4639 Morningside Drive  
Bloomington IN 47401, USA

**Abstract**— We briefly outline the use of higher-order splines and their reciprocal-bases in discretizing the volume-integral equations of electromagnetics. The discussion of higher-order splines, and the computation of their reciprocal-bases, is based on the theory presented in Chapters 4 and 5 of [1]. The discretization is carried out by means of the method of moments, in which the expansion functions are the higher-order splines, and the testing functions are the corresponding reciprocal-basis functions. Thus, the method is not Galerkin, but the structure of the resulting equations is quite regular, nevertheless.

## I. SPLINES BASED UPON HIGHER-ORDER CONVOLUTIONS OF THE UNIT PULSE

Consider a regular one-dimensional grid, whose spacing is  $h$ . Relative to this grid we define  $\pi(x)$  to be the unit pulse

$$\pi(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

and  $\pi_{m+1}(x)$  to be the  $m$ th-order convolution of  $\pi(x)$  (we define  $\pi_1(x) = \pi(x)$ ).

The higher-order convolutions of the unit pulse can be written in terms of the piecewise polynomials,  $\alpha_m^k(x)$ , as

$$\pi_{m+1}(x) = \sum_{k=0}^m \alpha_m^k(x-k) \theta(x-k),$$

where  $\theta(z) \equiv \pi(z)$  is the characteristic function of the unit interval, and the  $\alpha_m^k(x) = \sum_{j=0}^m a_m(k,j) x^j/j!$  are listed in Table 1.

Manuscript received July 7, 1991.

TABLE 1.

| Order (m) | $\alpha_m^k(x) = \sum_{j=0}^m a_m(k,j) x^j/j!$                                                                                                               |
|-----------|--------------------------------------------------------------------------------------------------------------------------------------------------------------|
| m=0       | $\alpha_0^0(x) = 1$                                                                                                                                          |
| m=1       | $\alpha_1^0(x) = x$<br>$\alpha_1^1(x) = 1 - x$                                                                                                               |
| m=2       | $\alpha_2^0(x) = x^2/2$<br>$\alpha_2^1(x) = 1/2 + x - x^2$<br>$\alpha_2^2(x) = 1/2 - x + x^2/2$                                                              |
| m=3       | $\alpha_3^0(x) = x^3/6$<br>$\alpha_3^1(x) = 1/6 + x/2 + x^2/2 - x^3/2$<br>$\alpha_3^2(x) = 4/6 - x^2 + x^3/2$<br>$\alpha_3^3(x) = 1/6 - x/2 + x^2/2 - x^3/6$ |

The coefficients  $a_m(k,j)$  satisfy the recursion relation

$$a_m(k,j) = a_{m-1}(k,j-1) - a_{m-1}(k-1,j-1) \\ 1 \leq k \leq m-1, 1 \leq j \leq m. \quad (2)$$

## II. RECIPROCAL-BASES FOR $\pi_{m+1}(x)$

Define  $\lambda_m(x)$  to be a function of compact support that will generate the reciprocal-basis system, just as  $\pi_{m+1}(x)$  generates the basis system.

We want

$$(h\pi_{m+1,h}^j, \lambda_{m,h}^k) = \delta_{jk}, \quad (3)$$

where  $\delta_{jk} = 1$ , if  $j = k$ ;  $= 0$ , if  $j \neq k$ , and

$$\pi_{m+1,h}^j = \frac{1}{h} \pi_{m+1}(x/h - j), \quad \lambda_{m,h}^k = \frac{1}{h} \lambda_m(x/h - k). \quad (4)$$

Upon substituting (4) into (3), we get

$$\frac{1}{h} \int \pi_{m+1}(x/h - j) \lambda_m(x/h - k) dx = \delta_{jk}. \quad (5)$$

Now let

$$\lambda_m(x) = \pi(x) \sum_{0 \leq l \leq m} b_l^{(m)} \frac{x^l}{l!}. \quad (6)$$

Then

$$\lambda_m(x/h - k) = \pi(x/h - k) \sum_{0 \leq l \leq m} b_l^{(m)} \frac{(x/h - k)^l}{l!}, \quad (7)$$

and we get

$$\sum_{0 \leq l \leq m} b_l^{(m)} \frac{1}{h} \int_{kh}^{(k+1)h} \pi_{m+1}(x/h - j) \frac{(x/h - k)^l}{l!} dx = \delta_{jk}. \quad (8)$$

Now let  $x/h - k = z \Rightarrow dx/h = dz$ ,  $x/h - j = z + k - j$ . Then

$$\sum_{0 \leq l \leq m} b_l^{(m)} \int_0^1 \pi_{m+1}(z + k - j) \frac{z^l}{l!} dz = \delta_{jk}. \quad (9)$$

We must determine the restriction of  $\pi_{m+1}(z + k - j)$  to the interval,  $(0, 1)$ :

$$\pi_{m+1}(z) = \sum_{n=0}^m \alpha_m^n(z - n) \theta(z - n), \quad (10)$$

where  $\theta(z)$  is the characteristic function of the unit interval. This implies that

$$\pi_{m+1}(z + k - j) = \sum_{n=0}^m \alpha_m^n(z + k - (j + n)) \theta(z + k - (j + n)). \quad (11)$$

Hence, because of the presence of the characteristic function,  $\theta(z + k - (j + n))$ , the term that survives in the interval  $(0, 1)$  is  $k = j + n$ ; i.e.,

$$\pi_{m+1}(z) = \alpha_m^{(k-j)}(z), \quad \text{for } 0 \leq z \leq 1. \quad (12)$$

Thus,

$$\sum_{0 \leq l \leq m} b_l^{(m)} \int_0^1 \alpha_m^{(k-j)}(z) \frac{z^l}{l!} dz = \delta_{jk}, \quad (13)$$

or,

$$\sum_{0 \leq l \leq m} b_l^{(m)} \int_0^1 \alpha_m^n(z) \frac{z^l}{l!} dz = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } 1 \leq n \leq m. \end{cases} \quad (14)$$

When we apply these results to the computation of the reciprocal-bases for the four lowest-order splines, we get

$$\begin{aligned} \lambda_3(x) &= \pi(x)(-184 + 2460x - 6600x^2 + 4620x^3) \\ \lambda_2(x) &= \pi(x)(15 - 108x + 120x^2) \\ \lambda_1(x) &= \pi(x)(-2 + 6x) \\ \lambda_0(x) &= \pi(x). \end{aligned} \quad (15) \quad \text{Then,}$$

For example, the reciprocal-basis for the third-order spline-basis is  $\lambda_3(x/h - k)$ .

Note that whereas the higher-order splines possess higher-order continuity properties, the reciprocal bases to these splines are all discontinuous.

### III. DISCRETIZING INTEGRAL EQUATIONS WITH HIGHER-ORDER SPLINES

The basic integral equation for the anomalous current produced by a flaw within a substrate is:

$$\mathbf{J}^{(i)}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) + \sigma(\mathbf{r}) \int_{\text{flaw}} \mathcal{G}(\mathbf{r}; \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') d\mathbf{r}', \quad (16)$$

where

$$\mathcal{G}(x - x', y - y'; z, z') = \iint_{-\infty}^{\infty} \tilde{\mathcal{G}}(k_x, k_y; z, z') e^{-j[k_x(x-x') + k_y(y-y')]} dk_x dk_y. \quad (17)$$

Thus

$$\begin{aligned} \mathbf{J}^{(i)}(\mathbf{r}) &= \mathbf{J}(\mathbf{r}) + \sigma(\mathbf{r}) \int dz' \iint_{-\infty}^{\infty} \tilde{\mathcal{G}}(k_x, k_y; z, z') \cdot \\ &\tilde{\mathbf{J}}(k_x, k_y; z') e^{-j[k_x(x-x') + k_y(y-y')]} dk_x dk_y. \end{aligned} \quad (18)$$

Introduce a regular, three-dimensional grid, whose spacing is  $\delta x$ ,  $\delta y$ , and  $\delta z$ , and expand  $\mathbf{J}(\mathbf{r})$  as the three-dimensional tensor-product of one-dimensional splines of arbitrary order,  $n$ , defined on this grid:

$$\mathbf{J}(\mathbf{r}) = \sum_l \sum_m \sum_k \mathbf{I}_{lmk} F_l^{(n)}\left(\frac{x}{\delta x}\right) F_m^{(n)}\left(\frac{y}{\delta y}\right) F_k^{(n)}\left(\frac{z}{\delta z}\right), \quad (19)$$

so that

$$\tilde{\mathbf{J}}(k_x, k_y; z) = \sum_l \sum_m \sum_k \mathbf{I}_{lmk} \tilde{F}_l^{(n)}(k_x) \tilde{F}_m^{(n)}(k_y) F_k^{(n)}\left(\frac{z}{\delta z}\right). \quad (20)$$

$$\begin{aligned} \mathbf{J}^{(i)}(\mathbf{r}) = & \sum_{l,m,k} \mathbf{I}_{lmk} F_l^{(n)}\left(\frac{x}{\delta x}\right) F_m^{(n)}\left(\frac{y}{\delta y}\right) F_k^{(n)}\left(\frac{z}{\delta z}\right) \\ & + \sum_{L,M,K} \sigma(\mathbf{r}) \iint_{-\infty}^{\infty} e^{-j[k_x x + k_y y]} \tilde{F}_L^{(n)}(k_x) \tilde{F}_M^{(n)}(k_y) \\ & \left\{ \int dz' \tilde{\mathcal{G}}(k_x, k_y; z, z') \cdot \mathbf{I}_{LMK} F_K^{(n)}\left(\frac{z'}{\delta z}\right) \right\} dk_x dk_y. \end{aligned} \quad (21)$$

Our notation is  $F_l^{(n)}(x/\delta x) = \pi_{n+1}(x/\delta x - l)$ .

Let  $\{\lambda_l^{(n)}(x/\delta x)\}$  be the reciprocal-basis to  $\{F_l^{(n)}(x/\delta x)\}$ . This means that

$$\int \lambda_l^{(n)}(x/\delta x) F_l^{(n)}(x/\delta x) dx/\delta x = \delta_{ll'}. \quad (22)$$

We are going to take moments of (21) with respect to the reciprocal-basis. Hence, we first multiply (21) by  $\lambda_{l'}^{(n)}(x/\delta x) \lambda_{m'}^{(n)}(y/\delta y) \lambda_{k'}^{(n)}(z/\delta z)$ , and then integrate. The result is

$$\mathbf{I}_{lmk}^{(i)} = \mathbf{I}_{lmk} + \sigma_{lmk} \sum_L \sum_M \sum_K \mathbf{G}_{kK}(l-L, m-M) \cdot \mathbf{I}_{LMK}, \quad (23)$$

where,

$$\begin{aligned} \mathbf{I}_{lmk}^{(i)} &= \frac{1}{\delta x \delta y \delta z} \iiint \mathbf{J}^{(i)}(\mathbf{r}) \lambda_l^{(n)}(x/\delta x) \lambda_m^{(n)}(y/\delta y) \lambda_k^{(n)}(z/\delta z) d\mathbf{r} \\ &= \frac{\sigma_{lmk}}{\delta x \delta y \delta z} \iiint F_l^{(n)}(x/\delta x) F_m^{(n)}(y/\delta y) F_k^{(n)}(z/\delta z) \mathbf{E}^{(i)}(\mathbf{r}) d\mathbf{r} \\ \sigma_{lmk} &= \frac{1}{\delta x \delta y \delta z} \iiint \sigma(\mathbf{r}) \left[ \lambda_l^{(n)}(x/\delta x) \right]^2 \left[ \lambda_m^{(n)}(y/\delta y) \right]^2 \left[ \lambda_k^{(n)}(z/\delta z) \right]^2 d\mathbf{r} \\ \mathbf{G}_{kK}(l-L, m-M) &= \iint_{-\infty}^{\infty} \tilde{F}_l^{*(n)}(k_x) \tilde{F}_L^{(n)}(k_x) \tilde{F}_m^{*(n)}(k_y) \tilde{F}_M^{(n)}(k_y) \tilde{\Gamma}_{kK}(k_x, k_y) dk_x dk_y \\ \tilde{\Gamma}_{kK}(k_x, k_y) &= \int dz \int dz' \tilde{\mathcal{G}}(k_x, k_y; z, z') F_k^{(n)}(z/\delta z) F_K^{(n)}(z'/\delta z). \end{aligned} \quad (24)$$

Note that

$$\tilde{F}_l^{*(n)}(k_x) \tilde{F}_L^{(n)}(k_x) = e^{jk_x(L-l)} \left( \frac{\sin(k_x/2)}{k_x/2} \right)^{2(n+1)},$$

which is a function of the difference of the two arguments,  $l$  and  $L$ . If  $n = 0$ , we get the expansion and testing in pulse-functions that were used in [2] and [3].

We have applied this theory to the case in which the expansion and testing functions are both pulses (the pulse functions are their own reciprocal bases) [2], [3]. Of course, this gives us the familiar Galerkin variant of the method of moments, and our results have been quite good. By using higher-order splines and their reciprocal bases in the manner suggested in this paper, we retain the same structure for the resulting integral equation as in the pulse-pulse situation, but, of course, the field solutions will be smoother.

There are several desirable features of the discretized integral equations that result from the approach of this paper. The coefficient of the first term on the right-hand side of (23) is always unity, regardless of the order of the expansion function. If we used a Galerkin approach, this coefficient would be a multi-diagonal matrix, depending upon the order of the splines used for expansion and testing. For example, the matrix would be tri-diagonal for first-order splines, penta-diagonal for second-order splines, etc.

A second feature is that we have a well-defined, optimal way of defining the discretized coefficient,  $\sigma_{lmk}$ , that appears in the integral equations. We do not need to use a "stair-step" approximation to the discretized region, or rely on an inaccurate *ad hoc*

method for defining the discretized material parameters. We have presented a consistent way of defining these parameters when used in conjunction with higher-order splines.

Similarly, the computation of the matrix elements,  $\mathbf{G}_{kK}(l-L, m-M)$ , is made quite systematic. One can use the same piece of code written for the

pulse-pulse combination, and substitute the necessary Fourier transforms for the higher-order splines.

We have used the method of higher-order splines and their reciprocal bases in discretizing transient boundary-integral equations, as well [4], [5]. In these papers we used one-dimensional second-order splines and their reciprocal bases to discretize the time-domain response of the space-time boundary-integral equation. The method produced stable, and accurate, results.

#### ACKNOWLEDGMENT

This work was supported by the Indiana Corporation for Science and Technology.

#### References

- [1] J. Aubin, *Approximation of Elliptic Boundary-Value Problems*, Wiley-Interscience, New York, 1972.
- [2] J. R. Bowler, L. D. Sabbagh, and H. A. Sabbagh, "A Theoretical and Computational Model of Eddy-Current Probes Incorporating Volume Integral and Conjugate Gradient Methods," *IEEE Trans. Magnetics*, Vol. 25, No. 3, May 1989, pp. 2650-2664.
- [3] H. A. Sabbagh, J. R. Bowler, and L. D. Sabbagh, "A Volume Integral Code for Eddy-Current Nondestructive Evaluation," *J. Applied Computational Electromagnetics Society*, Vol. 4, No. 1, Spring 1989, pp. 3-22.
- [4] S. Barkeshli, H. A. Sabbagh, D. J. Radecki, and D. W. Yergeau, "Analysis of Transient Electromagnetic Field Coupling to a Metallic Enclosure Using Boundary Integral/Finite Element Technique," *COMPEL*, Volume 9 (1990), Supplement A, pp. 87-90.
- [5] S. Barkeshli, H. A. Sabbagh, and D. J. Radecki, "A Novel Boundary Integral/Finite Element Method for the Analysis of Transient Electromagnetic Field Coupling to a Metallic Enclosure," 1990 *IEEE Antennas and Propagation Symposium Digest*, Volume 1, pp. 122-125.