

The Use of Parametric Elements in the Moment Method Solution of Static and Dynamic Volume Integral Equations

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Abstract—The use of two- and three-dimensional parametric curved subdomains (elements) in the numerical solution of electromagnetic problems formulated in terms of volume integro-differential equations is considered, and a technique to evaluate the static and dynamic moment integrals on these subdomains is presented. In the source region, the evaluation of the moment integrals is performed by separating the integrands into a singular and a regular part. The regular integrals are numerically evaluated in a parameter space where the integration domain assumes a very simple form. Then, it is shown how the singular integrals are always reducible to nonsingular boundary integrals by use of a nonlinear transformation for the integration variables and an analytic integration on one variable. The reduced singular integrals are then evaluated numerically in a second parameter space, where the boundary of the original curved element are expressed in a simple form. The use of matrix notation throughout the paper allows for very compact expressions which can be readily translated into computer programs. Two simple two-dimensional scattering problems in the frequency domain are considered to compare the results obtained using quadratic isoparametric curved elements against the results given by linear expansion functions on triangular elements. It is shown that the use of curved parametric elements permits a better geometrical description of the problem, a reduced number of unknowns, shorter computation times, and more accurate results with respect to the usually employed linear elements.

I. INTRODUCTION

IN THE NUMERICAL solution of electromagnetic problems formulated as volume integro-differential equations, one is frequently forced to use approximations of the geometries under investigation, as a result of the subdivision of the original domain into simple, smaller regions where the definition of expansion functions is made easier. In the recent literature, for instance, triangular subdomains have been used in the numerical solution of two-dimensional problems [1], whereas tetrahedral subdomains have been employed to model three-dimensional structures [2]. Indeed, up to now these subdomains have proven to be the most effective, owing to their general applicability. However, the drawback of these or similar subdomains is that the geometrical boundaries of the original region are modeled only by piecewise-linear approximations. Thus in order to approximate the original region with an almost equal volume model, one may either use more subdomains in the neighborhood of the boundaries or rescale to equal volume a coarser model after that subdivision has been done [2]. Theoretically, the

first solution always works, but in many practical cases it is so expensive that it can be adopted only to a certain extent; the latter is somewhat empirical, while its application to abrupt inhomogeneous regions is quite problematic, since different regions embedded into each other cannot be rescaled independently.

Thus, a problem of great interest in the method of moments application is to ensure that a small number of elements (subdomains) can represent a relatively complex form of the type which is liable to occur in real, rather than academic, problems. This problem has found a very effective solution in finite-element analysis through the development of parametric elements. Parametric (curved) elements are generated by distortion from simple forms (such as triangles, rectangles, tetrahedrons, rectangular prisms, etc.) so as to obtain other elements of more flexible shape which better match the object to be approximated [3]. To the author's knowledge, this is the first time that parametric elements are proposed in the method of moments analysis.

The major numerical difficulty encountered in the method of moments analysis is due to the singular kernel of the integral equations that renders the evaluation of the integrals in the source region particularly problematic, while the application of ordinary numerical integration techniques is straightforward when parametric elements are used in finite-element analysis, as only regular integrals occur there.

When parametric elements are used in method of moments applications, the definition of the basis functions is not problematic, since, as concisely illustrated in Section II, the natural solution to this problem is to adopt for these elements the same simple families of expansion functions used in finite-element analysis. Of course, when these basis functions are used, several possibilities are left open, since one may choose between subparametric, isoparametric, or superparametric elements (see again [3]). In the last two cases, a further advantage is due to the use of higher order expansion functions. In the procedure we discuss here, when the moment integral is evaluated in the source region, we only have to require the existence of a convenient interval around the observation point where the expansion functions may be approximated by the first two or three terms of a Taylor series. This assumption is valid provided that the shape distortion operated to obtain the parametric element is limited (i.e., does not lead to element twisting or warping). This rather obvious condition, which amounts to the existence of one-to-one mapping between points of the undistorted and distorted

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elements is readily obtained by considerations similar to the ones in [3, pp. 186–191]. However, since also in the finite-element method the practical application of parametric elements is only possible under this condition, the regularity of the expansion functions is assumed throughout the paper and not further discussed.

In order to evaluate the moment integrals in the source region, we separate the integrands into a singular and a regular part, and introduce two different transformations of variables. The first one is typical of finite-element analysis and is used to map a curved element into the “parent” (undistorted) element, in order to integrate numerically the regular part of the integrands. The parent element belongs to a parameter space and has a very simple shape. The second transformation, which is highly nonlinear and has no equivalent in the finite-element analysis, is only needed to integrate the singular part of the functions. This transformation, which is similar to the one proposed in [4], is used in order to reduce the singular part of the integrals to the integration of regular functions on the curved boundary of the element. These boundary integrals are then given directly in a form suitable for numerical integration in a parameter space, where the *curved boundary* of the elements are described in a simple way.

In Section II, the integration procedure for static and dynamic two-dimensional potential functions on parametric elements is explained. The integration of three-dimensional static and dynamic potential functions is illustrated in Section III.

Section IV is devoted to a comparison between linear and parametric elements in the solution of a two-dimensional frequency-domain scattering problem. Namely, the total field on the contour and inside a solid and a hollow penetrable circular cylinder at normal incidence are considered. The results obtained prove the usefulness of the theoretic results given here and suggest that parametric elements are superior also in the moment method solution of dynamic (and static) volume integral equations, since these elements permit a better geometrical description of the problem, a reduced number of unknowns, shorter computation times, and more accurate results than the ones provided by the commonly used linear elements.

II. THE INTEGRATION OF TWO-DIMENSIONAL POTENTIAL FUNCTIONS ON PARAMETRIC ELEMENTS

In two-dimensional static or quasi-static problems the kernel of the moment integrals is given in terms of a static Green function g_s , whereas for problems formulated in the frequency domain it is given in terms of a dynamic Green function g , where (see, e.g., [4], [5], [6])

$$g_s = \ln(|\rho - \rho'|) \quad \nabla_t g_s = \frac{\rho - \rho'}{|\rho - \rho'|^2} \quad (1)$$

$$g = -\frac{j}{4} H_0^{(2)}(\tau|\rho - \rho'|) \quad \nabla_t g = \frac{j}{4} \tau H_1^{(2)}(\tau|\rho - \rho'|) \frac{\rho - \rho'}{|\rho - \rho'|} \quad (2)$$

and ρ is the observation point; ρ' the source (integration) point; $H_0^{(2)}$, $H_1^{(2)}$ are the Hankel functions of the second kind

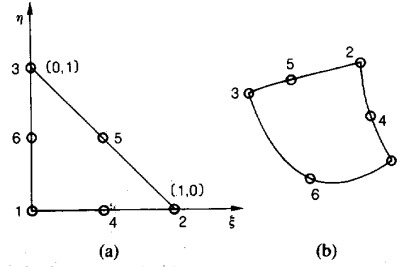


Fig. 1. Quadratic parametric distortion of a plane trilateral element. (a) Undistorted “parent” element in the local space (ξ , η). (b) Parametric element in the global space (x , y).

and of order zero and one, respectively; τ is the transverse wavenumber, and ∇_t the transverse gradient operator, operating on the unprimed coordinates. In the dynamic case, a time variation $\exp(j\omega t)$ is assumed and suppressed. These functions are associated with a two-dimensional source distribution $F(\rho')$, with which one may represent, when necessary, any vector component of a source field $\mathbf{F}(\rho')$. The problem is to evaluate on a given curved element the following potential integrals:

$$\iint F(\rho') \left(\frac{g_s}{g} \right) d\rho' \quad (3)$$

$$\iint F(\rho') \nabla_t \left(\frac{g_s}{g} \right) d\rho'. \quad (4)$$

It is convenient to evaluate (3), (4) by expressing their integrands as the sum of a singular part plus a regular one as follows:

$$F(\rho') \left(\frac{g_s}{g} \right) = [S_1 + R_1] \left(\frac{g_s}{g} \right) \quad (5)$$

$$F(\rho') \nabla_t \left(\frac{g_s}{g} \right) = [S_2 + R_2] \nabla_t \left(\frac{g_s}{g} \right) \quad (6)$$

where

$$S_1 = F(\rho) \quad R_1 = R_1(\rho') = F(\rho') - S_1 \quad (7a)$$

$$\lim_{\rho' \rightarrow \rho} [R_1 \left(\frac{g_s}{g} \right)] = 0 \quad (7b)$$

$$S_2 = S_2(\rho') = F(\rho) + [\nabla_t' F(\rho')] \cdot (\rho' - \rho) \quad \text{at } \rho' = \rho$$

$$R_2 = R_2(\rho') = F(\rho') - S_2(\rho') \quad (8a)$$

$$\lim_{\rho' \rightarrow \rho} [R_2 \nabla_t \left(\frac{g_s}{g} \right)] = 0. \quad (8b)$$

The Integration of the Regular Part

Without loss of generality, the integration procedure is here discussed for a trilateral parametric element of the (x , y) plane obtained by a quadratic distortion of the triangular “parent” defined in the local (ξ , η) space (see Fig. 1). The curved element is determined by giving the position in the (x , y) plane of six nodes, denoted by the subscripts (1, 6). The parent element is the right triangle $1 \equiv (0, 0)$; $2 \equiv (1, 0)$; $3 \equiv (0, 1)$. The other parent nodes (4, 6) are at the midpoints of the

triangle sides. The parametric transformation represented in Fig. 1 is

$$[x, y] = [N_1, N_2, N_3, N_4, N_5, N_6] \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_6 & y_6 \end{bmatrix} \quad (9)$$

where the "shape" functions N_i ($i = 1, 6$) are given in terms of the local area coordinates (L_1, L_2, L_3) as follows [3]:

$$\begin{aligned} N_1 &= L_1(2L_1 - 1) & N_2 &= L_2(2L_2 - 1) & N_3 &= L_3(2L_3 - 1) \\ N_4 &= 4L_1L_2 & N_5 &= 4L_2L_3 & N_6 &= 4L_1L_3 \end{aligned} \quad (10)$$

$$L_1 = 1 - \xi - \eta \quad L_2 = \xi \quad L_3 = \eta. \quad (11)$$

The Jacobian of the transformation (9) is

$$J(\xi, \eta) = \det \left\{ \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \dots & \frac{\partial N_6}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \dots & \frac{\partial N_6}{\partial \eta} \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_6 & y_6 \end{bmatrix} \right\}. \quad (12)$$

The well-known general condition for a *one-to-one* mapping is that the sign of the Jacobian of the transformation remains unchanged at all the points of the domain mapped. However, simpler rules to check the one-to-one mapping valid up to quadratic distortion are, for instance, given in [3], and with reference to this we may note that, as it happens in finite-element analysis, parabolic distortion in practice should suffice.

The contribution to the potential integrals (3), (4) of the regular parts of (5), (6) may be evaluated in the local space according to the following expressions:

$$\iint_{\text{parent domain}} J(\xi', \eta') R_1 \left(\frac{g_s}{g} \right) d\xi' d\eta' \quad (13)$$

$$\iint_{\text{parent domain}} J(\xi', \eta') R_2 \nabla_t \left(\frac{g_s}{g} \right) d\xi' d\eta'. \quad (14)$$

When the observation point is outside the parametric element, (3), (4) have no singular point, and one may substitute in (13), (14) to R_1 and R_2 the function $F(\rho')$. Numerical formulas for the approximate evaluation of double integrals such as (13), (14) are readily available in the literature (see, e.g., [3] and [7]). By use of (9) the samples of the functions g_s, g required by the adopted quadrature formula are readily evaluated in the (x, y) space for any given position (ξ', η') of the integration point.

It is now important to stress the convenience of using the

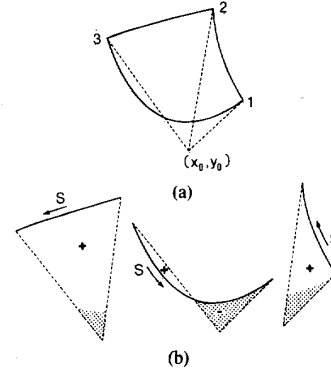


Fig. 2. The observation point (x_0, y_0) is used as a pivot to subdivide the integration domain into elementary subdomains. The curvilinear variable s , on each curved side of the element, is used to orient these subdomains. (a) The corner nodes of the element are joined by straight segments to the observation point. (b) On each subdomain, the sign of the Jacobian of the transformation (17) is indicated. The dotted regions are those external to the original element.

same families of expansion functions of the finite-element technique also in the proposed method of moments technique. The use of these basis functions is very appealing because they are directly defined in the local space (i.e., the (ξ, η) space) and in a simple way, and also because there is no limitation to their degree (in fact, quadratic and cubic expansion functions are quite common in finite-element analysis) [3]. Thus it is possible to achieve a given degree of approximation in the results with less elements of higher order and, actually, with a smaller number of unknowns [3] (see Section IV). Furthermore, in the procedure discussed here, we show how to evaluate moment integrals which involve the gradient of the Green functions. Thus one may use a point-matching technique (delta sampling at the nodal points of the elements) in building the system matrix for the problem under consideration, thereby avoiding the time-consuming second integration typical of numerical testing procedure such as the Galerkin one [8], [9]. Note also that, if the expansion functions are given in the local space, when point-matching at the nodal points of the element (i.e., the nodes of the parent), one does not need to invert the parametric transformation (e.g., (9)) to evaluate the singular and regular parts of the integrands required by (13) and (14) (cf. (7a), (8a)).

The Integration of the Singular Part

The surface integral of the singular part of potential functions on curved polygonal domains is always amenable to integration in the global space (i.e., the (x, y) plane) on trilateral subdomains which have only one curved side. These subdomains are obtained by joining with straight segments the observation point with each corner node of the curved polygonal domain. In Fig. 2(a), for instance, the observation point (x_0, y_0) placed outside the curved element previously considered yields the three nondegenerate trilateral subdomains of Fig. 2(b).

Furthermore, once the orientation of the element contour is chosen, for instance by counterclockwise ordering of the element nodes (the same was done also in [4]), the parametric

transformations (such as (9)) yield an *oriented* parametric representation for each of the curved sides of the trilateral subdomains. Thus the coordinates of the nodes on the curved side of the three subdomains of Fig. 2, are ordered as follows:

$$\begin{aligned} [\mathbf{X}_1, \mathbf{Y}_1] &= \begin{bmatrix} x_2, y_2 \\ x_5, y_5 \\ x_3, y_3 \end{bmatrix} \\ [\mathbf{X}_2, \mathbf{Y}_2] &= \begin{bmatrix} x_3, y_3 \\ x_6, y_6 \\ x_1, y_1 \end{bmatrix} \\ [\mathbf{X}_3, \mathbf{Y}_3] &= \begin{bmatrix} x_1, y_1 \\ x_4, y_4 \\ x_2, y_2 \end{bmatrix} \end{aligned} \quad (15)$$

and the parabolic parametric representation for the curved side of the i th ($i = 1, 3$) trilateral subdomain takes the form

$$\begin{aligned} [x, y]_i &= [h_{xi}(s), h_{yi}(s)] \\ &= [(s-1)(2s-1), -4s(s-1), s(2s-1)] \cdot [\mathbf{X}_i, \mathbf{Y}_i], \\ 0 &\leq s \leq 1 \end{aligned} \quad (16)$$

and is oriented in the direction of increasing s .

The i th trilateral subdomain is easily mapped into a square subdomain by use of the nonlinear transformation [4]:

$$\begin{cases} x = h_{xi}(s)(1-\lambda) + \lambda x_0, \\ y = h_{yi}(s)(1-\lambda) + \lambda y_0, \end{cases} \quad \{0 \leq s \leq 1; 0 \leq \lambda \leq 1\} \quad (17)$$

whose Jacobian $J_i(s, \lambda)$

$$J_i(s, \lambda) = (1-\lambda)J_i(s) \quad (18)$$

may be given in terms of an antisymmetric matrix using the following result:

$$J_i(s) = \begin{bmatrix} x_0 \\ \mathbf{X}_i \end{bmatrix}^+ \begin{bmatrix} 0 & 3-4s & 8s-4 & 1-4s \\ 4s-3 & 0 & 4(s-1)^2 & -4s^2+4s-1 \\ 4-8s & -4(s-1)^2 & 0 & 4s^2 \\ 4s-1 & 4s^2-4s+1 & -4s^2 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ \mathbf{Y}_i \end{bmatrix} \quad (19)$$

where the superscript $+$ indicates the transpose operation.

It is important to point out that all the Jacobians (18) always have a first-order zero at the observation point (i.e., at $\lambda = 1$), while the parabolic parametric representation (16) yields a zero of the second order when the observation point is on the curved edge (or on its extension) of the trilateral subdomain (at $s = p$)

$$\begin{aligned} x_0 &= h_{xi}(p) \quad y_0 = h_{yi}(p) \\ J_i(s) &= 4(s-p)^2 [\mathbf{X}_i]^+ \cdot \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \cdot [\mathbf{Y}_i]. \end{aligned} \quad (20)$$

In the particular case of a straight edge, and when the observation point is aligned with it, it is easily proven from (20) that the Jacobian is identically zero. This result is in agreement with [5], where it was first noted that the contribution from straight edges to the potential integrals considered here is zero if the observation point lies on the edge or on its extension. This brief discussion on the sign of the Jacobian of the suggested transformation is important to understand the sign of the contribution to the total integral due to each domain used to subdivide a given curved element, for arbitrary location of the observation point. For instance, we note in Fig. 2 how the region of the second subdomain where the Jacobian is negative corresponds to the area external to the global element, and considered by the other two subdomains. The contribution given by the first and third subdomains due to this external area is indeed canceled during the integration on the second subelement.

By use of the transformation (17), and after a first integration on λ , one gets the following expression for the static integral:

$$\iint S_1 g_s d\rho' = S_1/2 \sum_i \int_0^1 J_i(s') \{ \ln(d_i) - 1/2 \} ds'_i \quad (21)$$

where d_i is the distance from the observation point and the integration point of the i th curved edge

$$\begin{aligned} d_i &= d_i(s') = \sqrt{d_{xi}^2 + d_{yi}^2} \\ \begin{cases} d_{xi} = x_0 - h_{xi}(s') \\ d_{yi} = y_0 - h_{yi}(s') \end{cases} \end{aligned} \quad (22)$$

In the case of parabolic edges, (19) and (21) yield

$$\begin{aligned} \iint S_1 g_s d\rho' &= S_1/2 \sum_i \left\{ 1/6 [\mathbf{X}_i]^+ \cdot \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix} \cdot [\mathbf{Y}_i] + \int_0^1 J_i(s') \ln d_i ds'_i \right\}. \end{aligned} \quad (23)$$

Similarly, for the dynamic integral, one gets [4], [10], [11]:

$$\iint S_1 g \, d\rho' = -\frac{jS_1}{4\tau^2} \sum_i \int_0^1 \frac{J_i(s')}{d_i^2} \cdot [\tau d_i H_1^{(2)}(\tau d_i) - j2/\pi] \, ds_i'. \quad (24)$$

The following matrix representations for the vectors and the dyadics that appear in (6) and (8a) are readily obtained from (17):

$$\nabla_i g_s = \frac{d_{xi}\hat{x} + d_{yi}\hat{y}}{(1-\lambda)d_i^2} = \frac{\hat{d}_i}{(1-\lambda)d_i} \quad (25a)$$

$$\nabla_i g = j \frac{\tau}{4} H_1^{(2)}[(1-\lambda)\tau d_i] \hat{d}_i \quad (25b)$$

$$\hat{d}_i = \frac{1}{d_i} \begin{bmatrix} d_{xi} \\ d_{yi} \end{bmatrix}_{x,y} \quad (25c)$$

$$(\rho - \rho') \nabla_i g_s = \hat{\mathbf{A}} \quad (26a)$$

$$(\rho - \rho') \nabla_i g = j \frac{\tau}{4} (1-\lambda) d_i H_1^{(2)}[(1-\lambda)\tau d_i] \hat{\mathbf{A}} \quad (26b)$$

$$\hat{\mathbf{A}} = \frac{1}{d_i^2} \begin{bmatrix} d_{xi}^2 & d_{xi}d_{yi} \\ d_{xi}d_{yi} & d_{yi}^2 \end{bmatrix}_{x,y}. \quad (26c)$$

Let us recall the two auxiliary functions g_3, g_4 given in [4]:

$$\begin{aligned} z g_3(z) &= \int_0^z H_1^{(2)}(u) u \, du \\ &= (z\pi/2)[S_0(z)H_1^{(2)}(z) - S_1(z)H_0^{(2)}(z)] \end{aligned} \quad (27)$$

$$\begin{aligned} z^2 g_4(z) &= \int_0^z H_1^{(2)}(u) u^2 \, du \\ &= z^2 [H_2^{(2)}(z) - j4/(\pi z^2)] \end{aligned} \quad (28)$$

where S_0, S_1 are the Struve functions of order zero and one, respectively. After a first integration over λ , one gets [4], [10]:

$$\iint S_2 \nabla_i g_s \, d\rho' = \sum_i \int_0^1 J_i(s') \cdot \{(F/d_i)\hat{d}_i - 1/2\hat{\mathbf{A}} \cdot \nabla_i' F\} \, ds_i' \quad (29)$$

$$\iint S_2 \nabla_i g \, d\rho' = \frac{j}{4} \sum_i \int_0^1 J_i(s') \cdot \{(F/d_i)g_3(\tau d_i)\hat{d}_i - g_4(\tau d_i)\hat{\mathbf{A}} \cdot \nabla_i' F\} \, ds_i' \quad (30)$$

where the constant quantities F and $\nabla_i' F$, which are the values at the observation point of the functions $F(\rho')$ and $\nabla_i' F(\rho')$, respectively, are readily evaluated in the parameter space (ξ, η) [3]. Polynomial approximation of g_3 and g_4 for real argument are given in [4].

The contour integrals (21), (23), (24), (29), and (30) can be performed numerically, as they are never singular, even when the distance d_i is zero on their integration domains, which happens only when the observation point is on the edge (at $s =$

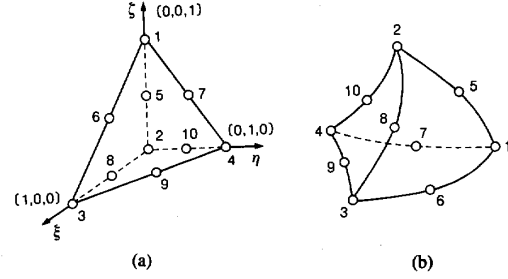


Fig. 3. Quadratic parametric distortion of a tetrahedral element. (a) Undistorted "parent" element in the local space (ξ, η, ζ) . (b) Parametric element in the global space (x, y, z) .

p). In such event, the zero of the Jacobian $J_i(s)$ (see, e.g., (20)) eliminates the singularity. Indeed, it is easily proved that the integrands of (21), (23), (24), (29), and (30) are zero in the limit $s \rightarrow p$ (i.e., $d_i \rightarrow 0$).

III. THE INTEGRATION OF THREE-DIMENSIONAL POTENTIAL FUNCTIONS ON PARAMETRIC ELEMENTS

Without loss of generality, the integration procedure for static and dynamic three-dimensional potential functions is commented on the curved tetrahedral element of Fig. 3, while the relevant static (G_s) and dynamic (G) Green functions are

$$G_s = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \quad \nabla G_s = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \quad (31)$$

$$G = \frac{\exp(-jK|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|}$$

$$\nabla G = -G \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} (1 + jK|\mathbf{r} - \mathbf{r}'|) \quad (32)$$

where $\mathbf{r}$ is the observation point, $\mathbf{r}'$ is the source (integration) point, ∇ operates on the unprimed coordinates, and $K = \omega\sqrt{\epsilon\mu}$ is the wavenumber. As in the previous two-dimensional analysis, in the dynamic case a time variation $\exp(j\omega t)$ is understood.

The four-faced element of the (x, y, z) space shown in Fig. 3(b) is obtained by a quadratic distortion of the tetrahedral "parent" of Fig. 3(a), defined in the local (ξ, η, ζ) space. The curved element is determined by giving the position in the (x, y, z) space of ten nodes (numbered from 1 to 10). The parent element is the tetrahedron $1 \equiv (0, 0, 1)$; $2 \equiv (0, 0, 0)$; $3 \equiv (1, 0, 0)$; $4 \equiv (0, 1, 0)$. As shown in Fig. 3(a), the other six parent nodes are located at the mid-points of the tetrahedron edges, and the parametric transformation at issue is as follows [3]:

$$[x, y, z] = [N_1, N_2, \dots, N_{10}] \cdot \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x_{10} & y_{10} & z_{10} \end{bmatrix} \quad (33)$$

where the shape functions N_i ($i = 1, 10$), in terms of the local

volume coordinates $(L_1, L_2, \dots, L_{10})$, are [3]

for corner nodes (i.e., nodes = 1, 4):

$$N_1 = (2L_1 - 1)L_1, \text{ etc.} \quad (34)$$

for midside nodes (i.e., nodes = 5, 10):

$$N_5 = 4L_1L_2, \text{ etc.} \quad (35)$$

with

$$L_1 = \xi \quad L_2 = 1 - (\xi + \eta + \zeta) \quad L_3 = \xi \quad L_4 = \eta. \quad (36)$$

The Jacobian of the transformation (33) is

$$J(\xi, \eta, \zeta) = \det \left\{ \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \dots & \frac{\partial N_{10}}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \dots & \frac{\partial N_{10}}{\partial \eta} \\ \frac{\partial N_1}{\partial \zeta} & \frac{\partial N_2}{\partial \zeta} & \dots & \frac{\partial N_{10}}{\partial \zeta} \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_{10} & y_{10} & z_{10} \end{bmatrix} \right\}. \quad (37)$$

The Green functions (31) and (32), when associated to a three-dimensional source distribution $F(\mathbf{r}')$, lead to the following representation in terms of a singular plus a regular functions:

$$F(\mathbf{r}') \begin{pmatrix} G_s \\ G \end{pmatrix} = [S + R] \begin{pmatrix} G_s \\ G \end{pmatrix} \quad (38a)$$

$$F(\mathbf{r}') \nabla \begin{pmatrix} G_s \\ G \end{pmatrix} = S \nabla \begin{pmatrix} G_s \\ G \end{pmatrix} + \begin{pmatrix} L_s \\ L \end{pmatrix} \quad (38b)$$

where

$$\begin{pmatrix} L_s \\ L \end{pmatrix} = R \nabla \begin{pmatrix} G_s \\ G \end{pmatrix} \quad (39)$$

$$S = S(\mathbf{r}') = F(\mathbf{r}) + [\nabla' F(\mathbf{r}')] \cdot (\mathbf{r}' - \mathbf{r}),$$

at $\mathbf{r}' = \mathbf{r}$

$$R = R(\mathbf{r}') = F(\mathbf{r}') - S(\mathbf{r}') \quad (40)$$

with

$$\lim_{\mathbf{r}' \rightarrow \mathbf{r}} \left[R \begin{pmatrix} G_s \\ G \end{pmatrix} \right] = 0. \quad (41)$$

The Integration of the Regular Part

As a consequence of the strong singularity in the gradient of the Green functions, L_s and L , though limited, are discontinuous at $\mathbf{r}' = \mathbf{r}$, so that a result similar to (8b) is not found in the three-dimensional case. As a matter of fact, while defining the function S (40), one may expand further the function $F(\mathbf{r}')$ to obtain L_s, L continuous functions, but then the integration of the singular parts would become extremely difficult. The contributions to the potential integrals due to the regular part of the integrands (38a) and (38b) are expressed in the local

space as follows:

$$\iiint_{\text{parent domain}} R \begin{pmatrix} G_s \\ G \end{pmatrix} J(\xi', \eta', \zeta') d\xi' d\eta' d\zeta' \quad (42)$$

$$\iiint_{\text{parent domain}} \begin{pmatrix} L_s \\ L \end{pmatrix} J(\xi', \eta', \zeta') d\xi' d\eta' d\zeta'. \quad (43)$$

The discontinuities of the functions L_s, L have to be resolved by the numerical integration algorithm, which anyway may be easily designed (or modified) to take care of this particular aspect.

The Integration of the Singular Part

The procedure for the integration of the singular parts on curved polyhedral domains is the extension of the one previously described for the two-dimensional case. By joining with straight segments the observation point with each corner node of the polyhedral element, the integration domain of the global space is subdivided into four-faced subelements, which have only one curved face (see Fig. 4). The parametric transformation (33) leads to an *oriented* parametric representation for each curved face of these subdomains. For the curved tetrahedral element of Fig. 3(b), the orientation of the subelements is in fact defined by counterclockwise ordering the nodes of the curved face with respect to the opposite corner node, i.e., with respect to the inward vector normal to the boundary curved surface (see also [4]) as follows.

Face 1

$$[\mathbf{X}_1, \mathbf{Y}_1, \mathbf{Z}_1] = \begin{bmatrix} x_2 & y_2 & z_2 \\ x_8 & y_8 & z_8 \\ x_3 & y_3 & z_3 \\ x_9 & y_9 & z_9 \\ x_4 & y_4 & z_4 \\ x_{10} & y_{10} & z_{10} \end{bmatrix}$$

Face 2

$$[\mathbf{X}_2, \mathbf{Y}_2, \mathbf{Z}_2] = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_7 & y_7 & z_7 \\ x_4 & y_4 & z_4 \\ x_9 & y_9 & z_9 \\ x_3 & y_3 & z_3 \\ x_6 & y_6 & z_6 \end{bmatrix}$$

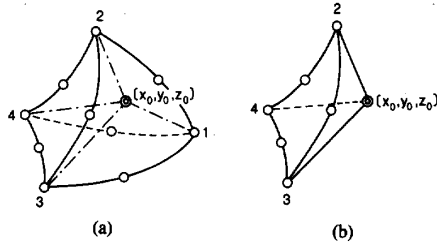


Fig. 4. The integration domain is subdivided into elementary subdomains. (a) The subdomains are obtained by joining with straight segments the observation point (x_0, y_0, z_0) to the corner nodes of the element. (b) Subdomain relative to face 1.

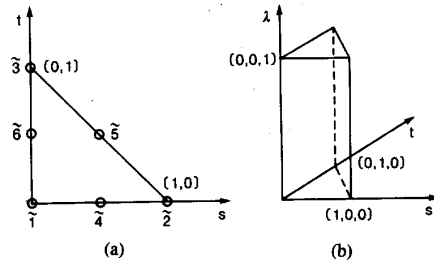


Fig. 5. The mapping of a tetrahedral subdomain into a simpler domain. (a) First, the trilateral curved face of the subdomain is mapped into a right triangle of the (s, t) space. (b) The subdomain is then mapped into the triangular prism of the space (s, t, λ) (see (47)).

Face 3

$$[\mathbf{X}_3, \mathbf{Y}_3, \mathbf{Z}_3] = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_5 & y_5 & z_5 \\ x_2 & y_2 & z_2 \\ x_{10} & y_{10} & z_{10} \\ x_4 & y_4 & z_4 \\ x_7 & y_7 & z_7 \end{bmatrix}$$

Face 4

$$[\mathbf{X}_4, \mathbf{Y}_4, \mathbf{Z}_4] = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_6 & y_6 & z_6 \\ x_3 & y_3 & z_3 \\ x_8 & y_8 & z_8 \\ x_2 & y_2 & z_2 \\ x_5 & y_5 & z_5 \end{bmatrix}$$

and the quadratic parametric representation for the curved face of each of the four subdomains of Fig. 4 takes the form

$$[x, y, z]_i = [h_{xi}(s, t), h_{yi}(s, t), h_{zi}(s, t)] \\ = [\tilde{N}_1, \tilde{N}_4, \tilde{N}_2, \tilde{N}_5, \tilde{N}_3, \tilde{N}_6] \cdot [\mathbf{X}_i, \mathbf{Y}_i, \mathbf{Z}_i] \quad (45) \quad \text{where}$$

the right triangle of Fig. 5(a):

$$\tilde{N}_k = (2\tilde{L}_k - 1)\tilde{L}_k, \quad k = 1, 3 \\ \tilde{N}_4 = 4\tilde{L}_1\tilde{L}_2 \quad \tilde{N}_5 = 4\tilde{L}_2\tilde{L}_3 \quad \tilde{N}_6 = 4\tilde{L}_1\tilde{L}_3 \\ \tilde{L}_1 = 1 - s - t \quad \tilde{L}_2 = s \quad \tilde{L}_3 = t. \quad (46)$$

The i th tetrahedral subdomain is then mapped into the triangular prism of Fig. 5(b) by use of the nonlinear transformation [4]:

$$\begin{cases} x = h_{xi}(s, t)(1 - \lambda) + \lambda x_0 \\ y = h_{yi}(s, t)(1 - \lambda) + \lambda y_0 \\ z = h_{zi}(s, t)(1 - \lambda) + \lambda z_0 \end{cases} \cdot \{ \{s \geq 0, t \geq 0, s + t \leq 1\}, 0 \leq \lambda \leq 1 \} \quad (47)$$

whose Jacobian is

$$J_i(s, t, \lambda) = (1 - \lambda)^2 J_i(s, t) \quad (48)$$

$$J_i(s, t) = \det \begin{bmatrix} \frac{\partial h_{xi}}{\partial s} & \frac{\partial h_{yi}}{\partial s} & \frac{\partial h_{zi}}{\partial s} \\ \frac{\partial h_{xi}}{\partial t} & \frac{\partial h_{yi}}{\partial t} & \frac{\partial h_{zi}}{\partial t} \\ x_0 - h_{xi} & y_0 - h_{yi} & z_0 - h_{zi} \end{bmatrix} \\ = \det \left\{ \begin{bmatrix} 1 & -\tilde{N}_1 & -\tilde{N}_4 & -\tilde{N}_2 & -\tilde{N}_5 & -\tilde{N}_3 & -\tilde{N}_6 \\ 0 & \frac{\partial \tilde{N}_1}{\partial s} & \frac{\partial \tilde{N}_4}{\partial s} & \frac{\partial \tilde{N}_2}{\partial s} & \frac{\partial \tilde{N}_5}{\partial s} & \frac{\partial \tilde{N}_3}{\partial s} & \frac{\partial \tilde{N}_6}{\partial s} \\ 0 & \frac{\partial \tilde{N}_1}{\partial t} & \frac{\partial \tilde{N}_4}{\partial t} & \frac{\partial \tilde{N}_2}{\partial t} & \frac{\partial \tilde{N}_5}{\partial t} & \frac{\partial \tilde{N}_3}{\partial t} & \frac{\partial \tilde{N}_6}{\partial t} \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ \mathbf{X}_i & \mathbf{Y}_i & \mathbf{Z}_i \end{bmatrix} \right\}. \quad (49)$$

where the shape functions $\tilde{N}_i$ ($i = 1, 6$) are defined in a new parameter space (s, t) by means of the local area coordinates of

Equation (49) expresses $J_i(s, t)$ as the determinant of the product of a "shape matrix" times an "observation-nodal

point matrix." The sum of all the elements in a row of the shape matrix is zero, so that it is immediately proven that in the particular case when the face $[X_i, Y_i, Z_i]$ is a plane, and for observation point lying in the same plane, the Jacobian of the nonlinear transformation (47) is identically zero. Furthermore, the Jacobians (48) have always a second-order zero at the observation point (i.e., at $\lambda = 1$), while we proved by algebraic formal computer manipulations that $J_i(s, t)$ and its first derivatives are always zero at the observation point, when this is on the quadratic curved face $[X_i, Y_i, Z_i]$ (or on its extension).

Let us call d_i the distance from observation point (x_0, y_0, z_0) and the integration point located on the i th curved face:

$$d_i = d_i(s', t') = \sqrt{d_{xi}^2 + d_{yi}^2 + d_{zi}^2} \quad (50)$$

$$\begin{cases} d_{xi} = x_0 - h_{xi}(s', t') \\ d_{yi} = y_0 - h_{yi}(s', t') \\ d_{zi} = z_0 - h_{zi}(s', t') \end{cases}$$

By introducing the normalized (complex) distance $\tilde{d}_i$:

$$\tilde{d}_i = -jKd_i \quad (51)$$

the transformation (47) yields

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = \hat{\mathbf{d}}_i, \quad \hat{\mathbf{d}}_i = \frac{d_{xi}\hat{x} + d_{yi}\hat{y} + d_{zi}\hat{z}}{d_i} \quad (52)$$

$$G = -jK \frac{\exp[(1-\lambda)\tilde{d}_i]}{(1-\lambda)\tilde{d}_i} \quad (53)$$

$$(\mathbf{r} - \mathbf{r}') \nabla G = -G[1 - (1-\lambda)\tilde{d}_i] \bar{\mathbf{A}},$$

$$\bar{\mathbf{A}} = \frac{1}{d_i^2} \begin{bmatrix} d_{xi}^2 & d_{xi}d_{yi} & d_{xi}d_{zi} \\ d_{xi}d_{yi} & d_{yi}^2 & d_{yi}d_{zi} \\ d_{xi}d_{zi} & d_{yi}d_{zi} & d_{zi}^2 \end{bmatrix} \quad (54)$$

A first integration on λ yields the following expressions for the dynamic integrals:

$$\iiint SG \, d\mathbf{r}' = \sum_i \iint J_i(s', t') \left\{ \frac{F(u_i + w_i)}{\tilde{d}_i} \frac{d_i}{\tilde{d}_i} + \frac{\nabla' F}{\tilde{d}_i} \hat{\mathbf{d}}_i \left[\frac{2(u_i + w_i)}{\tilde{d}_i} - w_i \right] \right\} ds'_i dt'_i \quad (55)$$

$$\iiint S \nabla G \, d\mathbf{r}' = \sum_i \iint \frac{J_i(s', t')}{d_i} \left\{ \frac{F}{d_i} ((2u_i + w_i)\tilde{d}_i) + \left[\frac{3(u_i + w_i)}{\tilde{d}_i} - w_i \right] \nabla' F \cdot \bar{\mathbf{A}} \right\} ds'_i dt'_i \quad (56)$$

where we set

$$w_i = \exp(\tilde{d}_i)$$

$$u_i = \frac{1 - w_i}{\tilde{d}_i} \quad (57)$$

In (55) and (56), the constant quantities F and $\nabla' F$ are the values at the observation point of the functions $F(\mathbf{r}')$ and $\nabla' F(\mathbf{r}')$, respectively; these constants are readily evaluated in the parameter space (ξ, η, ζ) [3].

In the limit of vanishing K , we obtain from (55) and (56) the results relative to the static integrals:

$$\iiint SG_s \, d\mathbf{r}' = \sum_i \iint J_i(s', t') \cdot \left[\frac{F}{2d_i} - \frac{\nabla' F}{3} \cdot \hat{\mathbf{d}}_i \right] ds'_i dt'_i \quad (58)$$

$$\iiint S \nabla G_s \, d\mathbf{r}' = - \sum_i \iint \frac{J_i(s', t')}{d_i} \cdot \left[\frac{F}{d_i} \hat{\mathbf{d}}_i - \frac{\nabla' F}{2} \cdot \bar{\mathbf{A}} \right] ds'_i dt'_i \quad (59)$$

IV. COMPARISON BETWEEN LINEAR ELEMENTS AND QUADRATIC ISOPARAMETRIC ELEMENTS IN THE SOLUTION OF THE TWO-DIMENSIONAL FREQUENCY-DOMAIN SCATTERING PROBLEM

To prove the practical usefulness of parametric elements, we developed a computer code, based on a point matching technique, to study the scattering problem from penetrable cylinders of arbitrary shape. At present, this code can deal with quadratic isoparametric elements having trilateral or quadrilateral shape. The expansion (i.e., shape) functions for the quadrilateral element are those given in [3, pp. 155, 160], whereas for the trilateral element these functions are those discussed in Section II((10), (11)). In this section some results from this code are compared against the ones obtained by a code developed by the author and described in [1], from which the present one is derived. All results presented are for homogeneous isotropic circular cylinder of radius a with relative permittivity $\epsilon_r = 4$, relative permeability $\mu_r = 1$. However, the code is currently used to study anisotropic, inhomogeneous, and lossy structures. The incident field considered in this section is a plane wave with electric field amplitude equal to unity that propagates normally to the axis of the cylinder (z -axis) in the $\hat{x}$ direction. The unknowns of the problem are the field components at the nodes of the cells where point matching is performed.

The first geometry considered is a solid cylinder (Fig. 6) with $K_0 a = 1.5$. In Figs. 6(a) and 6(b) the two models used by the isoparametric element program are reported. Fig. 6(c) shows the model used by the linear element program. Ten elements with 31 nodes and 18 elements with 55 nodes are defined in Figs. 6(a) and 6(b), respectively; whereas 98 triangles and 57 nodes are defined in Fig. 6(c). We note that the boundary of the scatterer is described exactly by parametric elements, whereas the linear element model results in a loss of area of the modeled scatterer and in a poorer approximation of its boundary. In Fig. 7 the computed results for the total E_z component along the contour of the cylinder are compared with the ones obtained by a Bessel series solution [12], [13], in case of TM ($E_x^i = E_y^i = H_z^i = 0$) plane wave normally incident at π . It can be noted that the use of 10 quadratic isoparametric elements with 31 nodes gives already better

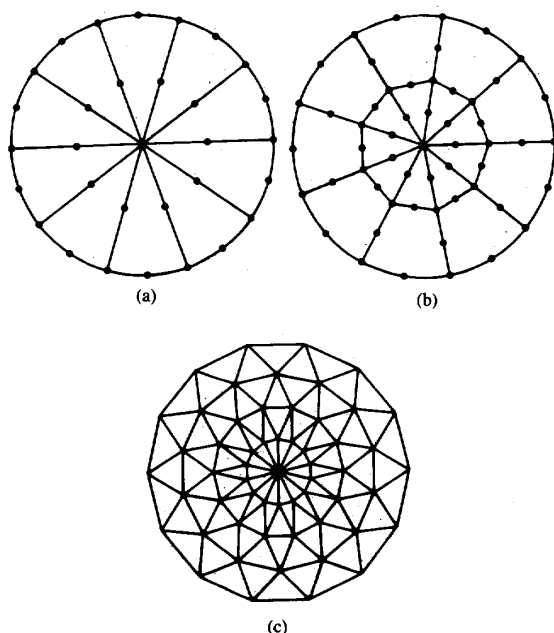


Fig. 6. Subdivision of the geometrical cross section of a solid circular cylinder (x is horizontal). (a) Quadratic parametric model with 10 elements and 31 nodes. (b) Quadratic parametric model with 18 elements and 55 nodes. (c) Linear element model with 98 triangles and 57 nodes.

results than the use of linear elements and 57 nodes. Since for both programs compared here the number of unknowns is proportional to the number of nodes, parametric (curved) elements give better results than linear elements using, in this case, almost half (i.e., 54 percent) of the unknowns needed by the linear element program. This fact results in a strong reduction of the computation time required to solve the linear system of equations (see Table I). Furthermore, the use of quadratic expansion functions for both real and imaginary parts of the field components allows for very accurate interpolation in magnitude and phase of the results known at the nodes of the scatterer, while the same is not true for linear elements. In Fig. 7 we also report the results relative to the isoparametric model with 55 nodes (Fig. 6(b)), in order to show the convergence of the results and to compare the computation time of the two programs for almost equal number of unknowns (see Table I).

The second geometry considered here is the hollow circular cylinder of Fig. 8 with thickness equal to $0.8a$ and $K_0a = 1$. Fig. 8(a) shows the isoparametric model defined by only 4 elements and 20 nodes, Fig. 8(b) shows the model used by the linear element program, defined by 84 triangles and 56 nodes. Both the external and the inner boundary of the scatterer are described almost exactly by parametric elements, while these are only approximated by the linear element model. In Fig. 9 the computed results for the total E_z component along the two contours of the cylinder are compared with the ones obtained by a Bessel series solution [12], [13], in case of TM plane wave normally incident at π . The computed results along the x -axis inside the cylinder are reported in Fig. 10. One may note that the isoparametric

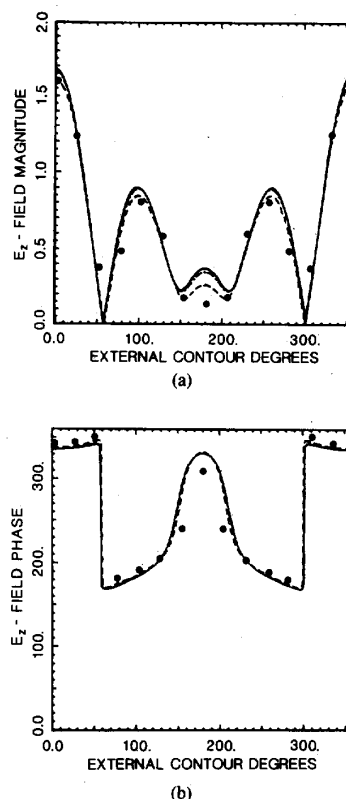


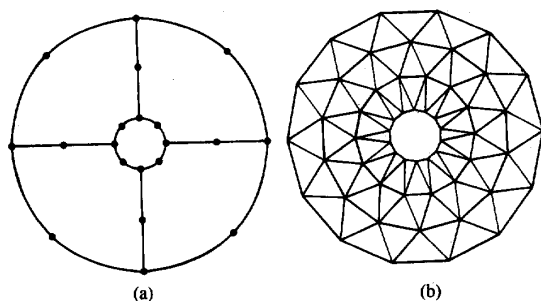
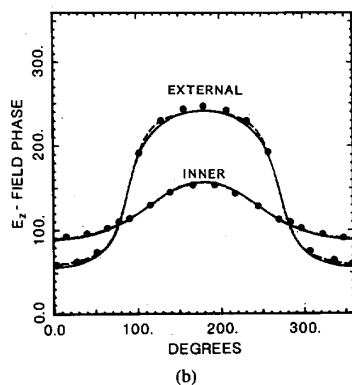
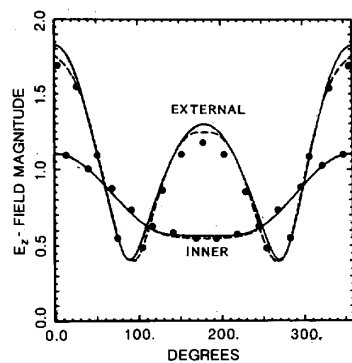
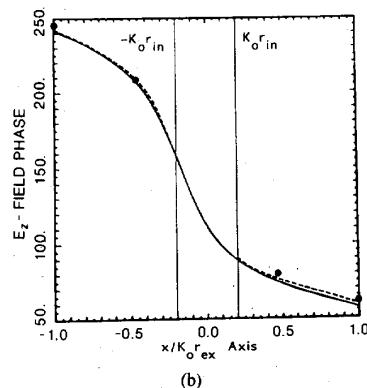
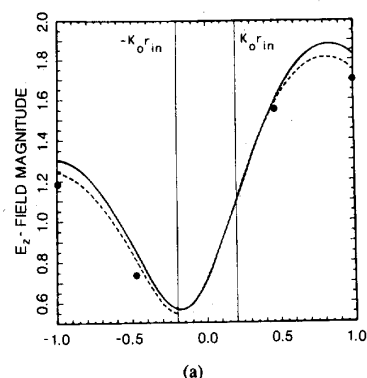
Fig. 7. (a) Amplitude and (b) phase of total E_z along the contour of the solid circular cylinder of Fig. 6 with radius a , for $K_0a = 1.5$, $\epsilon_r = 4$, $\mu_r = 1$. The solid line shows the result obtained by Bessel series solution, the dashed line results were obtained with the model of Fig. 6(a), the dash-dot results with the model of Fig. 6(b), and the dots indicate the result obtained with the linear element model (Fig. 6(c)).

model with 20 nodes gives better results than the 56-node linear model, using only 36 percent of the unknowns employed by the linear element program.

The computation times required to solve these scattering problems on a VAX 8800 are presented in Table I, where M is the matrix fill time, S the solution time for both TM and TE polarization, and T is the sum of M and S . The solution of the linear system was performed by Gauss elimination with total pivoting. The number of unknowns used in each run of the programs is six times the number of nodes, since the unknowns are the six field components at the nodes, even if, at normal incidence, it is possible to solve two problems of halved size instead of the complete problem [1]. The matrix fill time for the isoparametric element program is higher than that required by the linear element program, since the regular integrals on the quadrilateral parametric elements were performed by using a ninth-degree 20-point integration formula [7, p. 258] and on the trilateral parametric elements by a seventh-degree 16-point integration formula [7, p. 314]. The contour integrals (24) and (30) were performed by use of a three-point Gauss integration formula. The increase of the matrix fill time for the isoparametric element program with respect to that required by the linear element program is, however, rewarded by the higher degree of accuracy in the

TABLE I
COMPUTATION TIMES

Number of Triangles	Elements	Number of Nodes	Nodes on the Contour(s)	M (s)	S (s)	T (s)	
84	4	56	14 + 14	5.59	113.92	119.51	hollow cylinder
		20	8 + 8	8.33	4.5	12.83	
98	10	57	14	6.33	115.38	121.71	solid cylinder
	18	55	20	21.12	17.02	38.14	
			18	74.13	102.75	176.88	

Fig. 8. Subdivision of the geometrical cross section of a hollow circular cylinder (x is horizontal). (a) Quadratic parametric model with 4 elements and 20 nodes. (b) Linear element model with 84 triangles and 56 nodes.Fig. 9. (a) Amplitude and (b) phase of total E_z along the contours of the hollow circular cylinder of Fig. 8 with radius a and thickness $0.8a$, for $K_0 a = 1$, $\epsilon_r = 4$, $\mu_r = 1$. The solid line shows the result obtained by Bessel series solution, the dashed line results were obtained with the model of Fig. 8(a), and the dots indicate the result obtained with the linear element model (Fig. 8(b)).Fig. 10. (a) Amplitude and (b) phase of total E_z along the x -axis inside the hollow circular of Fig. 8 with radius a and thickness $0.8a$, for $K_0 a = 1$, $\epsilon_r = 4$, $\mu_r = 1$. The solid line shows the result obtained by Bessel series solution, the dashed line results were obtained with the model of Fig. 8(a), and the dots indicate the result obtained with the linear element model (Fig. 8(b)).

results and by a strong reduction of the solution time. Furthermore, it has to be noted that the parametric element program in its present implementation uses the above mentioned integration formulas also when the observation point is far away from the element considered. In such cases it is possible to resort to integration formulas of lower degree, so that the matrix fill time may be considerably reduced, especially when models with a relatively high number of elements are under investigation.

V. CONCLUSION

The use of two- and three-dimensional parametric curved elements in the method of moments solution of static and

dynamic volume integro-differential equations has been illustrated. For the sake of brevity, the theoretic discussion was confined to triangular and tetrahedral parametric subdomains with quadratic distortion. However, the procedure given can readily be extended both to other elements (rectangles, triangular or rectangular prisms, etc.) and to higher degree distortion (e.g., cubic). The kernel of the integrals considered were both the Green function and its gradient; this allows the evaluation of the moment system matrix by means of the point-matching technique also when using parametric subdomains. An example of the application of the formulas to the numerical solution of the two-dimensional electromagnetic scattering problem by the method of moments is given by having developed a computer program which uses quadratic isoparametric elements; the computed field distributions inside circular scatterers are found to be in excellent agreement with the exact results. The superiority of parametric elements in the method of moments solution of volume integral equations has been shown through a practical example, since these elements permit a better geometrical description of the problem, a reduced number of unknowns, shorter computation times, and results more accurate than the ones provided by the commonly used linear elements.

REFERENCES

- [1] R. D. Graglia and P. L. E. Uslenghi, "Electromagnetic scattering from anisotropic materials, Part II: Computer code and numerical results in two dimension," *IEEE Trans. Antennas Propagat.*, vol. AP-35, pp. 225-232, Feb. 1978.
- [2] D. H. Schaubert, D. R. Wilton, and A. W. Glisson, "A tetrahedral modeling method for electromagnetic scattering by arbitrarily shaped inhomogeneous dielectric bodies," *IEEE Trans. Antennas Propagat.*, vol. AP-32, pp. 77-85, Jan. 1984.
- [3] O. C. Zienkiewicz, *The Finite Element Method*, 3rd ed. New York: McGraw-Hill, 1977.
- [4] R. D. Graglia, "Static and dynamic potential integrals for linearly varying source distributions in two and three-dimensional problems," *IEEE Trans. Antennas Propagat.*, vol. AP-35, pp. 662-669, June 1987.
- [5] D. R. Wilton, S. M. Rao, A. W. Glisson, D. H. Schaubert, O. M. Al-Bundak, and C. M. Butler, "Potential integrals for uniform and linear source distributions on polygonal and polyhedral domains," *IEEE Trans. Antennas Propagat.*, vol. AP-32, pp. 276-281, Mar. 1984.
- [6] R. D. Graglia and P. L. E. Uslenghi, "Electromagnetic scattering from anisotropic materials, Part I: General theory," *IEEE Trans. Antennas Propagat.*, vol. AP-32, pp. 867-869, Aug. 1984.
- [7] A. H. Stroud, *Approximate Calculation of Multiple Integrals*. Englewood Cliffs, NJ: Prentice-Hall, 1971.
- [8] R. F. Harrington, *Field Computation by Moment Methods*. New York: Macmillan, 1968.
- [9] R. Mittra, Ed., *Computer Techniques for Electromagnetics*. New York: Pergamon, 1973.
- [10] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*. New York: Academic Press, 1980.
- [11] M. Abramowitz and A. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1972.
- [12] W. H. Eggiman, "Scattering of a plane wave on a ferrite cylinder at normal incidence," *IRE Trans. Microwave Theory Tech.*, vol. MTT-8, pp. 440-445, July 1960.
- [13] Y. Chow, "Scattering of electromagnetic waves by coaxial ferrite cylinder of different tensor permeabilities," *Appl. Sci. Res.*, sec. B, vol. 8, pp. 290-298, 1960.

Roberto D. Graglia (M'87), for a photograph and biography, please see page 669 of the June 1987 issue of this TRANSACTIONS.