

# A Multilevel Fast Multipole Algorithm for Solving 3D Volume Integral Equations of Electromagnetic Scattering<sup>\*</sup>

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## 1. Introduction

The volume integral equation techniques can be applied to calculate electromagnetic wave scattering from dielectric materials with inhomogeneous permittivity [1-4]. However if not accelerated by fast solvers, the volume integral equation technique is applicable to electrically small problems only since the number of unknowns increases very rapidly with the size of the dielectric objects. A useful acceleration technique is the Conjugate Gradient Fast Fourier Transform (CGFFT) method [2,3]. This method uses the toeplitz property of the three-dimensional Green's function. It performs the matrix-vector multiplication in the Fourier spectral domain. Hence, both the computation time and the memory requirement are reduced to  $O(N \log N)$ . Here  $N$  is the number of unknowns, and it is proportional to the number of grid points. In order to apply CGFFT, the entire rectangular box that encloses the object must be gridded.

Apparently this algorithm is not efficient in solving problems involving objects that have sparse dielectric distribution, such as a tree, a building in which there are empty spaces of air. These objects have small fractional volume of dielectric  $f_v$ . Here  $f_v$  is defined as the ratio of the dielectric volume and the volume of the smallest rectangular box that encloses the object. For example,  $f_v$  for a rectangular dielectric is 1.0, and that of a dielectric sphere is 0.52.

In this paper, we present a multilevel fast multipole algorithm (MLFMA) [7-8] to accelerate the solution of volume integral equation. The advantage of this algorithm is that it achieves the computational complexity of  $O(N \log N)$  for dielectric objects of arbitrary  $f_v$  values.

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The basic concept of the fast multipole method (FMM) and its multilevel implementation have been reported elsewhere [5-8]. In this paper, we present the formulations and implementations that are specific to the volume integral equation. The discussion is in the frequency domain, and the time factor of  $\exp\{-i\omega t\}$  has been suppressed from the equations.

## 2. Near-Neighbor Elements

A volume integral equation (VIE) is formulated by writing the total electric field in the dielectric region as the sum of the scattered field and the incident field. For non-magnetic material, the VIE can be written as

$$\bar{E}(\bar{r}) = \int_V \bar{\bar{G}}(\bar{r}, \bar{r}') \cdot \bar{J}(\bar{r}') d\bar{r}' + \bar{E}^{inc}(\bar{r}), \quad \bar{r} \in V, \quad (1)$$

where the dyadic Green's function  $\bar{\bar{G}}(\bar{r}, \bar{r}') = i\omega\mu_b(\bar{I} + \nabla\nabla/k_b^2)g(\bar{r}, \bar{r}')$ , with  $g(\bar{r}, \bar{r}') = \exp\{ik_b|\bar{r} - \bar{r}'|\}/(4\pi|\bar{r} - \bar{r}'|)$ , and  $k_b$  being the wavenumber of the background material.

Since the equivalent volume current  $\bar{J}$  in (1) is related to the total electric field by  $\bar{J} = i\omega(\epsilon_b - \epsilon)\bar{E}$ , there is only one unknown vector function  $\bar{E}$  in equation (1).

The method of moments is then applied to convert equation (1) into a matrix equation. To this end, the dielectric volume is discretized into small volume cells  $V_j, j=1, 2, \dots, N$ , and the volume current in (1) is represented by  $\bar{J}(\bar{r}) = \sum_{j=1}^N a_j \chi(\bar{r}) \bar{f}_j(\bar{r})$ . Here  $a_j$  is the expansion coefficient,  $\chi = (\epsilon_b - \epsilon)/\epsilon$ , and  $\bar{f}_j(\bar{r})$  is the basis function. Several basis functions (such as the pulse basis [1,2], roof-top basis [3], and the 3D Rao-Wilton-Glison basis [4]) have been used for this purpose. In this paper, the 3D RWG basis on tetrahedrons is used because a set of tetrahedrons can represent 3D objects of arbitrary shapes. Let the testing function and the basis function be denoted by  $\bar{t}_i$  and  $\bar{f}_j$ , respectively, the general matrix element  $A_{ij}$  can then be written as

$$A_{ij} = \int_{V_i} \bar{t}_i(\bar{r}) \cdot \int_{V_j} \bar{\bar{G}}(\bar{r}, \bar{r}') \cdot \chi \bar{f}_j(\bar{r}') d\bar{r}' d\bar{r}. \quad (2)$$

A basic idea in the fast multipole method is to express the matrix element  $A_{ij}$  by different representations according to the distance,  $d_{ij}$ , between the basis function  $\bar{f}_j$  and the testing function  $\bar{t}_i$ . If  $d_{ij}$  is small,  $A_{ij}$  is called near-neighbor elements. Otherwise  $A_{ij}$  is called non-near-neighbor element. For near-neighbor elements,  $A_{ij}$  is calculated by the similar equations as given by [4]. To adequately process the singular integrals, the operator in (1) has been converted to the mixed potential representation. Hence,

$$A_{ij} = i\omega\mu_b \int_{V_i} \bar{t}_i(\bar{r}) \cdot \left[ \int_{V_j} g(\bar{r}, \bar{r}') \chi \bar{f}_j(\bar{r}') d\bar{r}' + \frac{\nabla}{k_b^2} \int_{V_j} g(\bar{r}, \bar{r}') \nabla' (\chi \bar{f}_j(\bar{r}')) d\bar{r}' \right] d\bar{r}. \quad (3)$$

### 3. Non-Near-Neighbor Elements

In FMM, the testing and basis functions are collected into groups. The efficiency in FMM is gained by calculating the interactions between well-separated testing and basis functions with an alternative expression. This alternative expression is based on the addition theorem of spherical wave functions. We let the testing function  $\bar{t}_i$  belong to group  $m$ , and the basis function  $\bar{f}_j$  belong to group  $n$ . If groups  $m$  and  $n$  are well separated (generally the separation is by at least one group of the same size), then the matrix element associated with  $\bar{t}_i$  and  $\bar{f}_j$  can be written as [8],

$$A_{ij} = \eta_0 \left( \frac{ik_b}{4\pi} \right)^2 \int d^2\hat{k} \bar{V}_{fmi}(\hat{k}) \cdot T_{mn}(\hat{k}) \cdot \bar{V}_{snj}(\hat{k}), \quad (4)$$

where the integration over  $\hat{k}$  is on a unit sphere, and

$$\bar{V}_{fmi}(\hat{k}) = \int_{V_i} (\bar{I} - \hat{k}\hat{k}) \cdot \bar{t}_i(\bar{r}) e^{i\bar{k} \cdot (\bar{r} - \bar{r}_m)} d\bar{r}, \quad (5)$$

$$T_{mn}(\hat{k}) = \sum_{l=0}^L i^l (2l+1) h_l^{(1)}(kr_{mn}) P_l(\hat{k} \cdot \hat{r}_{mn}), \quad (6)$$

$$\bar{V}_{snj}(\hat{k}) = \int_{V_j} (\bar{I} - \hat{k}\hat{k}) \cdot \bar{f}_j(\bar{r}) \chi(\bar{r}) e^{-i\bar{k} \cdot (\bar{r} - \bar{r}_n)} d\bar{r}. \quad (7)$$

In the above,  $\bar{r}_m$  and  $\bar{r}_n$  are the center position of group  $m$  and group  $n$ , respectively,  $\bar{r}_{mn} = \bar{r}_m - \bar{r}_n$ , and the selection of  $L$  can be found in [6].

### 4. Numerical Results

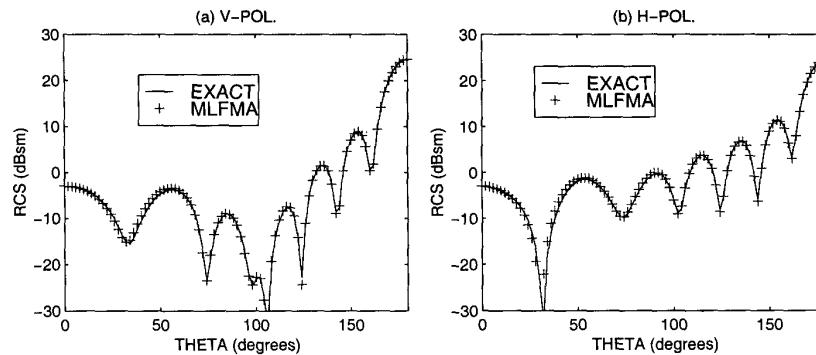
A numerical example is presented in this section to demonstrate the implementation of the multilevel fast multipole algorithm (MLFMA) for solving the volume integral equation of electromagnetic scattering. As shown in Figure 1, the RCS calculated using the MLFMA is compared with the exact solution for a spherical dielectric shell. The shell has inner and outer radii of  $1.5\lambda_0$  and  $1.6\lambda_0$ , respectively. The dielectric constant is  $\epsilon_r = 1.5 + 0.5i$ . The incident wave frequency is 300 MHz. The shell is discretized into 7056 tetrahedrons with 16464 unknowns. The average edge length of the tetrahedrons is  $0.1753\lambda_0$ . The Conjugate Gradient (CG) method is used to solve the matrix equation iteratively. The memory requirement for this problem is 152 MB by a two-level MLFMA, and the CPU time per iteration is 5.06 sec. on a SUN Ultra5 workstation.

### 5. Conclusion

The multilevel fast multipole algorithm has been applied for solving volume integral equation. Tetrahedron volume cells are used to discretize the material objects, providing a flexible representation for complex object shapes. Due to the reduction of the computational complexity, the method presented in this paper can be applied to solve electromagnetic scattering problems with electrically larger dielectrics.

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**Figure 1.** Comparison of the RCS for a spherical dielectric shell. The shell has inner and outer radii of  $1.5\lambda_0$  and  $1.6\lambda_0$ , respectively. The material permittivity is  $\epsilon_r = 1.5 + 0.5i$  and the incident wave is  $\theta = 0^\circ$ . Incident wave frequency is  $f = 300$  MHz. (a) Vertical polarization, (b) Horizontal polarization.