

Multilevel Fast Multipole Method Solution of Volume Integral Equations Using Parametric Geometry Modeling

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1 Introduction

Field computations for problems involving inhomogeneous materials have mostly been confined to either analytical approximations for canonical problem geometries or differential equation solver, such as the finite element (FE) method [1]. Integral equation formulations for inhomogeneous materials have not been exploited due to their high computational cost [2, 3, 4]. However, recent developments in fast algorithms, such as the multilevel fast multipole method (MLFMM) [5, 6] have enabled direct method of moments (MoM) solutions of volumetric integral equation formulations.

This paper outlines the MLFMM solution of a volume integral equation for scattering by arbitrarily shaped inhomogeneous dielectric structures. The approach uses three-dimensional conformal parametric subdomains to ensure fidelity of the geometrical representation. This low complexity computer code can be used in various areas of applied computational electromagnetics, ranging from microwave circuit simulations to remote sensing applications.

2 Volume Integral Equation Formulation

The electric-field-integral-equation (EFIE) for modeling the volume of a dielectric structure is given by

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \int_v d\mathbf{r}' \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot (k^2(\mathbf{r}') - k_0^2) \mathbf{E}(\mathbf{r}'), \quad (1)$$

where v is the domain of the dielectric volume, $\mathbf{E}_{inc}$ is the incident or excitation electric field, $k(\mathbf{r}) = k_0 \sqrt{\epsilon_r(\mathbf{r})}$ is the wavenumber inside the inhomogeneous medium, $\mathbf{r}$ is the position vector, k_0 is the free space wavenumber, and $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ denotes the dyadic Green's function (an $e^{-i\omega t}$ time dependence is assumed and suppressed). To solve for $\mathbf{E}$, we proceed to discretize the volume v using curvilinear hexahedral finite elements and introduce the expansion

$$\mathbf{E}(\mathbf{r}) \approx \sum_{i=1}^N a_i \mathbf{e}_i(\mathbf{r}). \quad (2)$$

where a_i are the unknown expansion coefficients, N denotes the number of basis functions used to discretize the domain and $\mathbf{e}_i(\mathbf{r})$ are the basis functions defined within the hexahedra and are given below.

Following standard MoM formulation, by substituting (2) into (1) and employing Galerkin's testing the integral equation is converted into the matrix system $\mathbf{Z}\mathbf{a} = \mathbf{b}$.

3 Geometry Modeling and Basis Functions

Referring to Fig. 1 (a), any point inside the hexahedron is a parametric mapping of a corresponding point in a unit cube through the transformation

$$\mathbf{r}(u, v, w) = \sum_{i=0}^2 \sum_{j=0}^2 \sum_{k=0}^2 \mathbf{r}_{ijk} L_{ijk}(u, v, w), \quad (u, v, w) \in ([0, 1], [0, 1], [0, 1]) \quad (3)$$

where $\mathbf{r}_{ijk}$ define the 27 points of the hexahedron and $L_{ijk}(u, v, w)$ are the quadratic Lagrange interpolation functions in three parameters (u, v, w) . These coefficients can be constructed using the 27 constraints $\mathbf{r}(0.0, 0.0, 0.0) = \mathbf{r}_{000}$, $\mathbf{r}(0.5, 0.0, 0.0) = \mathbf{r}_{100}$, $\mathbf{r}(1.0, 0.0, 0.0) = \mathbf{r}_{200}$, ..., $\mathbf{r}(1.0, 1.0, 1.0) = \mathbf{r}_{222}$.

3.1 Electric Field Basis Functions

The basis functions used in this work are a generalization of those given in [2]. We define three basis functions in each hexahedron in terms of the parametric tangent vectors as

$$\mathbf{e}^{(u)}(\mathbf{r}) = \frac{1}{\sqrt{G}} \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{e}^{(v)}(\mathbf{r}) = \frac{1}{\sqrt{G}} \frac{\partial \mathbf{r}}{\partial v}, \quad \mathbf{e}^{(w)}(\mathbf{r}) = \frac{1}{\sqrt{G}} \frac{\partial \mathbf{r}}{\partial w}. \quad (4)$$

Above, the determinant of the parametric transformation (3) is given by

$$G = \begin{vmatrix} g_{uu} & g_{uv} & g_{uw} \\ g_{vu} & g_{vv} & g_{vw} \\ g_{wu} & g_{wv} & g_{ww} \end{vmatrix}, \quad g_{\eta\xi} = \frac{\partial \mathbf{r}}{\partial \eta} \cdot \frac{\partial \mathbf{r}}{\partial \xi}. \quad (5)$$

Here, η and ξ represent any of the parameters u , v , and w . As expected, the basis functions defined here reduce to those in [2].

4 Multilevel Fast Multipole Method

MLFMM is used to reduce the $O(N^2)$ complexity of the matrix-vector product and the $O(N^2)$ memory requirements down to $O(N \log N)$. The low complexity and memory are achieved by employing spherical wave expansion of the free-space Green's function appearing in the MoM formulation. This enables a modified representation of the MoM matrix elements, which can be used to cast the matrix-vector product into a form that can be computed faster [5]. On applying a nested grouping strategy, it is possible to achieve the aforementioned algorithmic complexities [6]. In our implementation, we use the *k-means* algorithm [7]. This general grouping strategy enables the development of a very flexible multilevel grouping set-up. In particular, the clustering starts by forming 2 groups of the unknowns using the *k-means* algorithm. These groups constitute the clustering scheme for the 1st level. Subsequently, the *k-means* algorithm is used to generate 2 subgroups for each of the groups in the 1st level. Hence, 4 groups are generated for the 2nd level. Likewise, for the 3rd level, 2 subgroups are formed for each of these 4 groups. This grouping strategy is carried out successively until at least one of the smallest groups contains only a single member or element and thus further subdivision is not possible. This grouping set-up constitutes the backbone of the MLFMM algorithm. Crucial MLFMM parameters that determine the speed and the memory requirements of the algorithm (i.e. the number of multipoles and the size of the near-field matrix) are a function of this grouping process. For the examples given here, we formed

our multilevel grouping by dividing each group into 2 subgroups. However, this is not a necessity in the k-means algorithm, and one can use any number of subgroups at any level of the grouping set-up. This feature will prove to be useful when dealing with small details concentrated in a small region of an electrically large structure.

The number of terms kept in the multipole expansion is computed using the formula [5]: $L = kD + \ln(kD + \pi)$. A 2^{nd} order polynomial interpolation is used to generate the interpolation/interpolation matrices [6] to account for the different number of multipoles used within different levels.

5 Results and Conclusions

To verify and validate the proposed procedure for solving volume integral equations, we first considered the scattering of a 4λ diameter spherical shell of thickness 0.2λ and having $\epsilon_r = 2.75 + 0.3i$. Fig. 1 (b) shows the two principal cuts for the bistatic radar cross section as computed by the volumetric MLFMM and the Mie solution. The shell geometry is discretized using 24642 hexahedra (3 concentric hexahedron layers are used), resulting in a dense 73926×73926 MoM system. This discretization corresponds to about 15 elements per free-space wavelength (about 9 elements per wavelength inside the dielectric). The problem was solved on a 400 MHz SGI Origin 2000 computer with an available memory of 2 GBytes. Comparatively, the MoM solution would have required 43.7 GBytes to store the MoM matrix.

In Fig. 2 (a) we provide the complexity and the memory requirement of the MLFMM implementation. Also, Fig. 2 (b) compares the error in the finite element-boundary integral (FE-BI) and the volume integral equation (VIE) solutions for scattering by a dielectric sphere ($\epsilon_r = 4.0$) as a function of the number of unknowns. As seen, the VIE has a better error performance, e. g. for a 5% RCS error the VIE requires 250 unknowns whereas the FE-BI requires 1000 unknowns. It is also important to note that the convergence of the iterative solution of the MLFMM-VIE algorithm is superior to that of the FE-BI method. This is due to the fact that the VIE is a 2^{nd} kind integral equation, generating a strong diagonal in the resulting MoM matrix.

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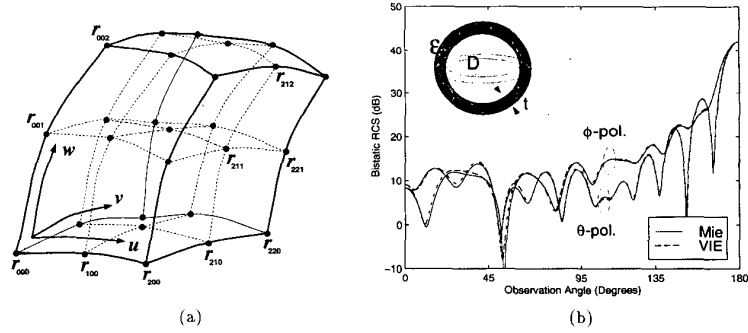


Figure 1: (a) Geometrical construction of a curvilinear hexahedral finite element, (b) Bistatic RCS of a 4λ diameter 0.2λ thick dielectric shell with $\epsilon_r \approx 2.75 + 0.3i$.

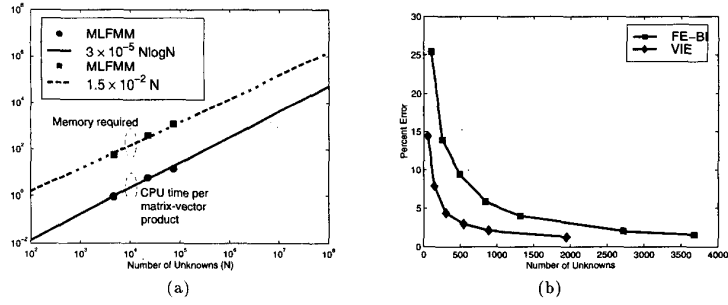


Figure 2: (a) Computational complexities MLFMM algorithm, (b) Convergence comparison of FE-BI and VIE methods.