

A New Iterative Method to Solve a Volume Integral Equation

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Abstract— A new iterative method to solve a volume integral equation is developed. The method consists in the division of the domain in subdomains, every subdomains is characterized by a moment matrix that determines the field induced by the excitation. The entire solution is built by interaction among the subdomains, with the incident field as the initial excitation.

1. Introduction

The computational of electromagnetic fields in a three-dimensional problem is very often solved using the moment method, which transform the integral equation into a system of algebraic equations. The major limiting factor of the moment method is its need for a large computer memory to store the matrix involved in the computation. An important quality is that, when we have the inverse matrix, the scattering fields due to any new excitation are calculated simply by multiplication of the inverse matrix by the new source vector.

Iterative methods can solve problems involving a larger number of unknowns than the moment method but when we have new excitation all process have to begin again. A typical example is in [1] where an iterative conjugate gradient method is used to solve a three-dimensional volume integral equation. That algorithm is an extension of [2], a two-dimensional conjugate gradient scheme.

This work proposes an iterative method to solve a volume integral equation. The method consists in the division of the domain in subdomains, every subdomain is characterized by a moment matrix that determines the field induced by an excitation. The entire solution is built by interaction among the subdomains with the incident field as the initial excitation. When we have the new excitation it is not necessary to calculate the moment matrices again, it is only necessary to start the calculations with the new initial excitation.

2. Formulation of the problem

Figure 1 shows a three-dimensional arbitrarily shaped heterogeneous dielectric body illuminated by a plane wave E^i . The problem can be formulated by replacing

the material body occupying V by an equivalent volume current J as shown.

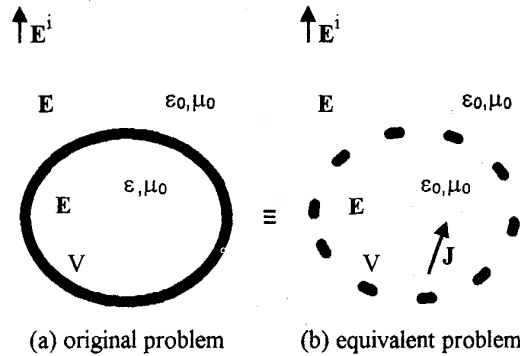


Fig. 1. Replacement of an heterogeneous dielectric body by an equivalent volume current.

A volume integral equation can be written as follows [3]

$$\frac{J(r)}{j\omega(\epsilon - \epsilon_0)} + j\omega A(r) + \nabla\phi(r) = E^i \quad (1)$$

where $j = \sqrt{-1}$ and

$$A(r) = \mu_0 \int_V J(r') \frac{e^{-jk_0|r-r'|}}{4\pi|r-r'|} dv' \quad (2)$$

$$\phi(r) = \frac{1}{\epsilon_0} \int_V \rho(r') \frac{e^{-jk_0|r-r'|}}{4\pi|r-r'|} dv' \quad (3)$$

Here ϵ_0 and ϵ are the permittivity of the free space and of the body, respectively. $\kappa_0^2 = \omega^2 \epsilon_0 \mu_0$ where ω is the angular frequency and μ_0 is the permeability of the free space. The charge density $\rho(r)$ is related to the equivalent current J by

$$\nabla \cdot J(r) = -j\omega \rho(r) \quad (4)$$

3. Discretization by tetrahedral cells

The solution of the integral equation (1) begins with the process of discretizing the volume V with

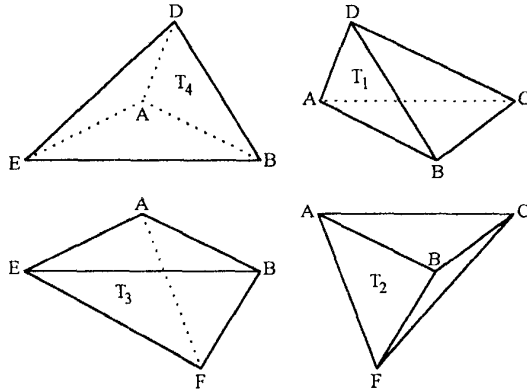
tetrahedral cells, generally of different cell sizes. The equivalence current in the body is expanded by the finite series

$$\mathbf{J} = \sum_{i=1}^N \alpha_i \mathbf{J}_i \quad (5)$$

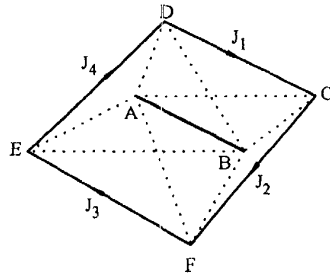
where $\mathbf{J}_i, i = 1, \dots, N$, represents a set of basis functions. These functions are defined, as reported in [4], by

$$\mathbf{J}_{AB} = \frac{\overline{XY}}{V_i} \text{ in the volume } V_i \text{ of tetrahedron } T_i \quad (6)$$

where $\overline{XY}$, which denotes the vector pointing from node X to node Y , can take the values $\overline{DC}, \overline{CF}, \overline{FE}$ and $\overline{ED}$, for $i=1, \dots, 4$ respectively; T_i denotes the i -th tetrahedron linked to edge AB and V_i the volume of the i -th tetrahedron (see Fig.2). The function is zero everywhere else. It is easy to show that (6) represents a solenoidal basis function [4]. When AB represents an edge on the surface of the dielectric, there is no closed circulation of the basis function around the edge. This ensures that the basis function is solenoidal only inside the dielectric object where one should not observe the spurious charges.



(a) volume elements associated with edge AB



(b) basis function associated with edge AB

Fig. 2 Base function defined on tetrahedrons

Substituting (5) in (1) and taking the inner product of the result with the function $\mathbf{J}_j, j = 1, \dots, N$, (Galerkin's method), one obtain

$$\sum_{i=1}^N \alpha_i \left[\left\langle \frac{\mathbf{J}_i}{j\omega(\epsilon - \epsilon_0)}, \mathbf{J}_j \right\rangle + \langle \mathbf{E}^s, \mathbf{J}_j \rangle \right] = \langle \mathbf{E}^i, \mathbf{J}_j \rangle, \quad j = 1, \dots, N \quad (7)$$

This equation represents a linear system of algebraic equations that can be solved by using any numerical method to solve linear systems. In terms of the equivalent currents, the scattered field takes the form

$$\mathbf{E}^s = j\omega \mathbf{A}(\mathbf{r}) + \nabla \phi(\mathbf{r}) \quad (8)$$

In the (7) the inner product is defined by

$$\langle \mathbf{B}, \mathbf{J}_j \rangle = \int_V \mathbf{B} \cdot \mathbf{J}_j \, dv' \quad (9)$$

4. The iterative method

We assume that the set of basis functions is defined so that it may be divided into M subsets. Each subset is defined in a region of the body, so the current expansion can be rewritten as

$$\mathbf{J} = \sum_{m=1}^M \sum_{i=1}^{N^m} \alpha_i \mathbf{J}_i \quad (10)$$

where N^m is the number of basis functions in the region M . The problem domain Ω is formed by union of M nonoverlapping subdomains Ω_i , so that

$$\Omega = \bigcup_{m=1}^M \Omega_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j \quad (11)$$

where the Ω_i represent the body's subdomains. Substituting (10) in (1) one obtain

$$\sum_{m=1}^M \left[\sum_{i=1}^{N^m} \alpha_i \frac{\mathbf{J}_i}{j\omega(\epsilon - \epsilon_0)} + \sum_{i=1}^{N^m} j\omega \alpha_i \mathbf{A}(\mathbf{r}) + \sum_{i=1}^{N^m} \alpha_i \nabla \phi(\mathbf{r}) \right] = \mathbf{E}^i$$

This equation may be written like a system of M coupled equations as follows

$$\sum_{i=1}^{N^1} \alpha_i \left[\frac{\mathbf{J}_i}{j\omega(\epsilon - \epsilon_0)} + j\omega \mathbf{A} + \nabla \phi \right] = \mathbf{E}^i - \sum_{m \neq 1}^M \mathbf{E}_{\text{rad}}^m \quad \text{in } \Omega_1$$

$$\begin{aligned} \sum_{i=1}^{N^2} \alpha_i \left[\frac{\mathbf{J}_i}{j\omega(\epsilon - \epsilon_0)} + j\omega \mathbf{A} + \nabla \phi \right] &= \mathbf{E}^i - \sum_{m \neq 2}^M \mathbf{E}_{\text{rad}}^m \quad \text{in } \Omega_2 \\ &\vdots \\ \sum_{i=1}^{N^M} \alpha_i \left[\frac{\mathbf{J}_i}{j\omega(\epsilon - \epsilon_0)} + j\omega \mathbf{A} + \nabla \phi \right] &= \mathbf{E}^i - \sum_{m \neq M}^M \mathbf{E}_{\text{rad}}^m \quad \text{in } \Omega_M \end{aligned} \quad (12)$$

$\mathbf{E}_{\text{rad}}^m$ is the field radiated by the current of subdomain Ω_m and is defined by

$$\mathbf{E}_{\text{rad}}^m = \sum_{i=1}^{N^m} j\omega \alpha_i \mathbf{A}(\mathbf{r}) + \sum_{i=1}^{N^m} \alpha_i \nabla \phi(\mathbf{r}) \quad (13)$$

We apply the moment method in (12) to define a linear system for each subdomain, so we have

$$\begin{aligned} \sum_{i=1}^{N^1} \alpha_i \langle \mathbf{L}(\mathbf{J}_i), \mathbf{J}_j \rangle &= \langle \mathbf{E}^i, \mathbf{J}_j \rangle - \sum_{m \neq 1}^M \langle \mathbf{E}_{\text{rad}}^m, \mathbf{J}_j \rangle \quad j=1, \dots, N^1 \quad \text{in } \Omega_1 \\ \sum_{i=1}^{N^2} \alpha_i \langle \mathbf{L}(\mathbf{J}_i), \mathbf{J}_j \rangle &= \langle \mathbf{E}^i, \mathbf{J}_j \rangle - \sum_{m \neq 2}^M \langle \mathbf{E}_{\text{rad}}^m, \mathbf{J}_j \rangle \quad j=1, \dots, N^2 \quad \text{in } \Omega_2 \\ &\vdots \\ \sum_{i=1}^{N^M} \alpha_i \langle \mathbf{L}(\mathbf{J}_i), \mathbf{J}_j \rangle &= \langle \mathbf{E}^i, \mathbf{J}_j \rangle - \sum_{m \neq M}^M \langle \mathbf{E}_{\text{rad}}^m, \mathbf{J}_j \rangle \quad j=1, \dots, N^M \quad \text{in } \Omega_M \end{aligned} \quad (14)$$

where $\mathbf{L}(\mathbf{J}_i) \equiv \frac{\mathbf{J}_i}{j\omega(\epsilon - \epsilon_0)} + j\omega \mathbf{A}(\mathbf{J}_i) + \nabla \phi(\mathbf{J}_i)$.

Rewriting (14) in matrix form we have

$$\begin{aligned} [Z_1] [I_1] &= [V_1] \\ [Z_2] [I_2] &= [V_2] \\ &\vdots \\ [Z_M] [I_M] &= [V_M] \end{aligned} \quad (15)$$

where $[I_t]$ is the vector of the unknown expansion coefficients and $[V_t]$ is the iterative excitation vector, both with respect to subdomain $[\Omega_t]$. From (14) $[I_t]$ and $[V_t]$ are given by

$$[I_t] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N_t} \end{bmatrix} \quad (16)$$

$$[V_t] = \begin{bmatrix} \langle \mathbf{E}^i, \mathbf{J}_1 \rangle - \sum_{m \neq t}^M \langle \mathbf{E}_{\text{rad}}^m, \mathbf{J}_1 \rangle \\ \langle \mathbf{E}^i, \mathbf{J}_2 \rangle - \sum_{m \neq t}^M \langle \mathbf{E}_{\text{rad}}^m, \mathbf{J}_2 \rangle \\ \vdots \\ \langle \mathbf{E}^i, \mathbf{J}_{N_t} \rangle - \sum_{m \neq t}^M \langle \mathbf{E}_{\text{rad}}^m, \mathbf{J}_{N_t} \rangle \end{bmatrix} \quad (17)$$

The elements of the moment matrix $[Z_t]$, in the subdomain $[\Omega_t]$, are given by

$$Z_{ji} = \langle \mathbf{L}(\mathbf{J}_i), \mathbf{J}_j \rangle \quad i, j = 1, \dots, N_t \quad (18)$$

The iterative solution to the linear system (15) is given by

$$\begin{aligned} [I_1]^n &= [Z_1]^{-1} [V_1]^n \\ [I_2]^n &= [Z_2]^{-1} [V_2]^n \\ &\vdots \\ [I_M]^n &= [Z_M]^{-1} [V_M]^n \end{aligned} \quad (19)$$

where the n superscript denotes the number of the iteration. As the vector $[I_t]$ is computed we use it immediately in the calculation of next vector. The convergence criterion is defined by

$$\|I_m^n - I_m^{n-1}\|_2 \leq \text{dist} \quad (20)$$

where $\|\mathbf{x}\|_2$ is the Euclidean norm given by

$$\|\mathbf{x}\|_2 = \left(\sum_{k=1}^K |x_k|^2 \right)^{1/2} \quad (21)$$

The physical significance of iterative method is that we are representing of body by M subdomains, each subdomain is excited by the incident field and by a secondary field radiated by the other subdomains so the entire solution is built by interaction among the subdomains with the moment matrix of each subdomain giving its response to the excitation.

5. Numerical results

In order to validate the method, we calculate the far field for a dielectric cube of sidelength $0.2\lambda_0$ (Fig. 3), where λ_0 is the free space wavelength. The cube was partitioned lengthwise into 512 cubes of equal size. Using pulse basis functions we have three unknowns

per cube, so it results in 1536 unknowns for the cube. To use the solenoidal basis functions we partition the cube with 384 tetrahedrons. The set of basis functions generated by these tetrahedrons does not form a linearly independent set. The kind of dependence among the functions will depend on the discretization. We eliminate the dependent functions building the Gram matrix [5] and removing the functions that generate linearly dependent rows in the matrix. In this case we have 480 linearly independent functions.

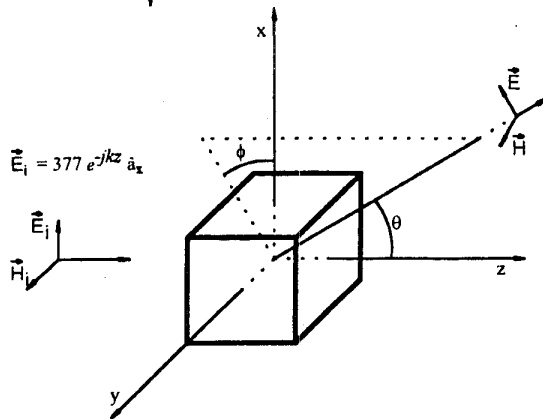


Fig. 3 Dielectric cube of sidelength $0.2 \lambda_0$

Figures 4 and 6 show the far field for a dielectric cube with $\epsilon = 4.0\epsilon_0$ and $\epsilon = 9.0\epsilon_0$, respectively. These results are in good agreement with those given in [6] and with the results calculated using the program of [7].

To apply the iterative method we divide the cube in 8 subdomains, each discretized with 48 tetrahedra that generate 72 linearly independent functions. The convergence criterion was set with $\text{dist} = 10^{-8}$ and was obtained with 15 and 41 iterations for the cube with $\epsilon = 4.0\epsilon_0$ and $\epsilon = 9.0\epsilon_0$, respectively. In the figures 5 and 7 it is shown the comparison of the far fields obtained with the direct and iterative method. It can be seen that an excellent agreement was obtained.

6. Conclusions

This paper has presented a new iterative method to solve a volume integral equation. It was used to analyze the scattering by a homogeneous cube and the results are in excellent agreement with those obtained using the direct method.

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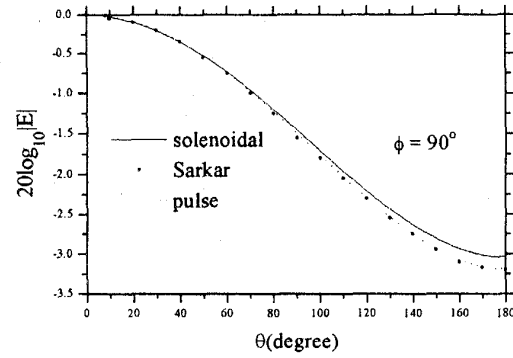
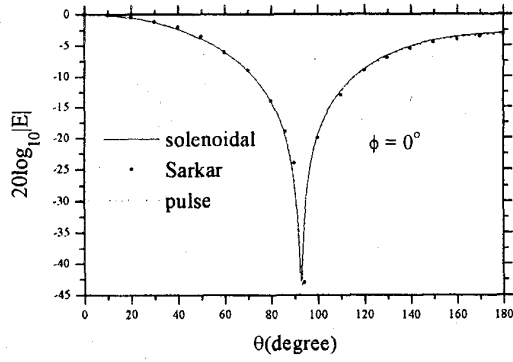


Fig. 4 Scattered field by a dielectric cube with $\epsilon = 4.0\epsilon_0$

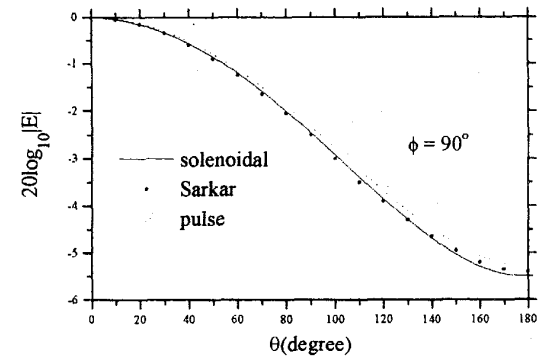
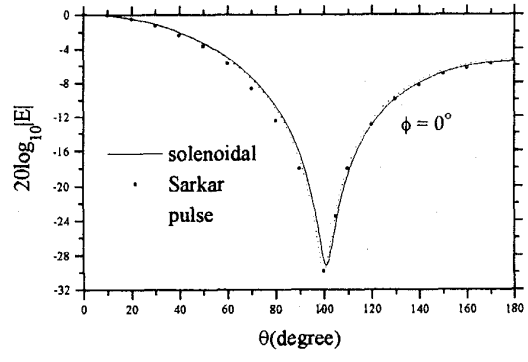


Fig. 6 Scattered field by a dielectric cube with $\epsilon = 9.0\epsilon_0$

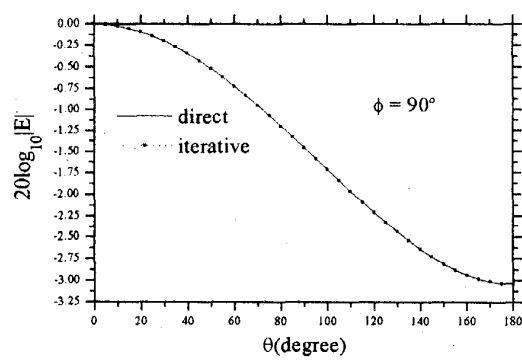
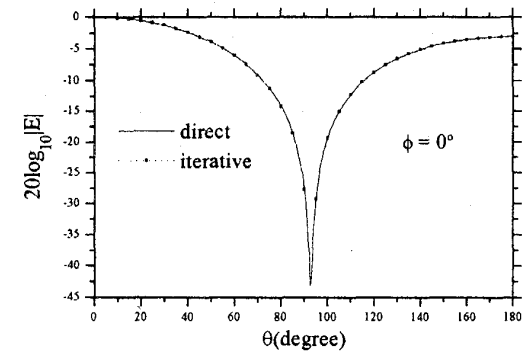


Fig. 5 Scattered field by direct and iterative methods dielectric cube with $\epsilon = 4.0\epsilon_0$

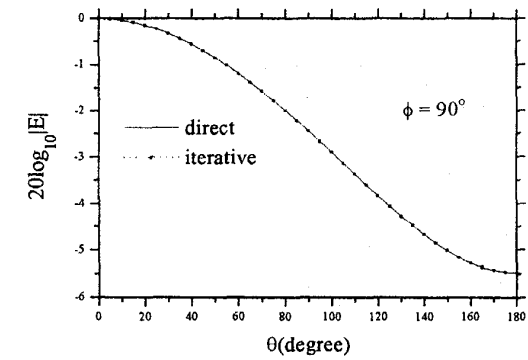
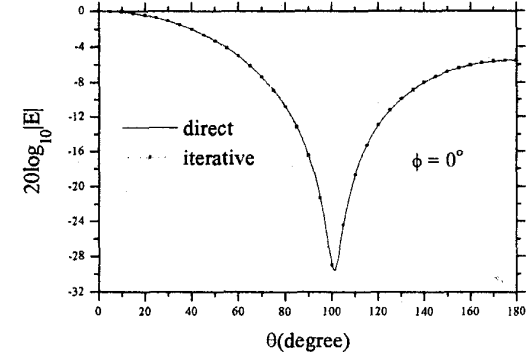


Fig. 7 Scattered field by direct and iterative methods dielectric cube with $\epsilon = 9.0\epsilon_0$