

Magnetic Field Computation in Superconducting Thin Film Structures by a Volume Integral Equation Method

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Abstract— We present a method to calculate the current density distribution, the magnetic field distribution and the inductance matrix of three dimensional superconducting structures by minimizing the free energy. In the case of high- T_c superconductors, the kinetic inductance becomes substantial and an accurate calculation of inductances is required. The London penetration depth λ and the kinetic inductance are included. The results of a program which realizes this method agree well with measurements of high- T_c dc-SQUIDS (superconducting quantum interference devices).

I. INTRODUCTION

Since the discovery of high temperature superconductivity in 1986, a lot of applications have been developed. Thin film technology is commonly used in order to produce superconducting circuits, rf-devices and high sensitive magnetic field sensors. The thicknesses of the superconducting films and the London penetration depth are in the same range of ten to several hundred nanometers.

Because the superconducting electrons move in a superconductor without any dissipation, their kinetic energy leads to a contribution to the total inductance known as the kinetic inductance. The geometrical inductance is determined by the magnetic field distribution in the whole space. In small structures with dimensions of a few micrometers, the kinetic inductance can be of the same order of magnitude as the geometrical inductance. For that reason, a special field calculation is necessary to obtain accurate results in the case of superconducting electrodes.

In some simple structures, such as superconducting microstrip lines [1], [2] and large square washers [3], the inductance can be obtained by analytical approximations.

The inductances of two-dimensional structures can be calculated by the Finite Difference Method [4]. However, the real field problem has open boundaries, whereas the Finite Difference Method requires boundary conditions in

a chosen area. As the solution area is limited, some errors can occur.

In a two-dimensional method of minimizing the free energy [5] only the superconductor has to be discretized. The field solution is not limited by any boundaries. With an enhancement of this method, the dissipation in the superconductor can be calculated [6].

In 1989, the Surface Current Method was presented [7]. With circulating currents on the surface of a three dimensional structure, the current density distribution and the inductances were calculated. The inductance matrix was obtained via the energy. The problem with this method is the approximation of the magnetic field near the line currents.

In [8], a method based on the Biot-Savart law to calculate the current density distribution and the inductances has been presented. In this method, the direction of the line currents has to be assumed.

In this paper we present a volume integral equation method based on Maxwell's equations and the London theory. Using the solution for the energy between two small cuboids with a given current density, the total energy of a structure has been calculated. By minimizing the total energy over the vector of the current distribution, a linear system of equations has been obtained. The solution to this system of equations yields the current distribution. Therefore, only the superconductor must be discretized. By variation of the current stimulation in several loops, the inductance matrix can be calculated. Furthermore, the magnetic field can be calculated in several planes. During the computation, the discretization is adapted automatically until the inductance matrix elements converge.

II. THEORETICAL BACKGROUND

The total energy of a superconducting structure can be written in common terms:

$$W = W_{kin} + W_m + W_f, \quad (1)$$

where W_{kin} , W_m and W_f are the kinetic energy, the magnetic field energy and the field energy of an external homogeneous magnetic field, respectively. The kinetic energy can be obtained with use of the differential equations of

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the London theory

$$\begin{aligned}\nabla \times (\Lambda \vec{J}_s) &= -\vec{B}, \\ \frac{\partial (\Lambda \vec{J}_s)}{\partial t} &= \vec{E},\end{aligned}\quad (2)$$

where $\Lambda = m_s/n_s e_s^2$, $\vec{J}_s$ is the supercurrent density, $\vec{B}$ is the induction, $\vec{E}$ is the electric field, m_s is the mass of the Cooper pairs, e_s is the charge of the Cooper pairs, and n_s is the density of the Cooper pairs. Using the kinetic energy of a single Cooper pair $m_s v^2/2$, the energy density w can be written as

$$w = n_s \frac{m_s}{2} v^2, \quad (3)$$

where v is the velocity of the Cooper pairs. Considering a cuboid with $J_s = \text{constant}$ over the cross section, the velocity can be expressed in the form

$$v = \frac{J_s}{n_s e_s}. \quad (4)$$

The energy density of the Cooper pairs is thus given by:

$$w = \frac{1}{2} \frac{m_s J_s^2}{n_s e_s^2} = \frac{1}{2} \Lambda J_s^2 = \frac{1}{2} \mu_0 \lambda^2 J_s^2, \quad (5)$$

where $\Lambda = \mu_0 \lambda^2$ results from the solution of London's equations in the superconducting half space. The kinetic energy is then obtained by integration over the volume V :

$$W_k = \int_V w dV = \frac{\mu_0}{2} \int_V \lambda^2 J_s^2 dV. \quad (6)$$

The magnetic energy W_m of the entire system can be written as:

$$W_m = \frac{1}{2} \int_V \vec{H} \cdot \vec{B} dV, \quad (7)$$

where $\vec{H}$ is the magnetic field. With the definition of the vector potential $\vec{A}_p$

$$\vec{B} = \nabla \times \vec{A}_p \quad (8)$$

and the vector identity

$$(\nabla \times \vec{A}_p) \cdot \vec{H} = \nabla \cdot (\vec{A}_p \times \vec{H}) + \vec{A}_p \cdot (\nabla \times \vec{H}) \quad (9)$$

the magnetic energy is obtained by:

$$W_m = \frac{1}{2} \int_A (\vec{A}_p \times \vec{H}) \cdot d\vec{A} + \frac{1}{2} \int_V \vec{A}_p \cdot (\nabla \times \vec{H}) dV. \quad (10)$$

For an integration area far away, the first term $\frac{1}{2} \int_A (\vec{A}_p \times \vec{H}) \cdot d\vec{A}$ is zero because the magnetic field $\vec{H}$ approaches zero faster than the surface area of the sphere increases [9]. Using Maxwell's equation in the stationary case

$$\vec{J} = \nabla \times \vec{H} \quad (11)$$

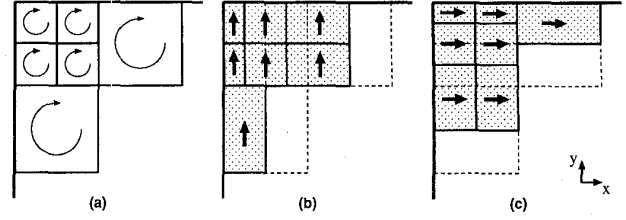


Fig. 1. Discretization: (a) basic mesh with circulating currents, (b) cuboids for currents in y-direction, (c) cuboids for currents in x-direction

the magnetic energy becomes:

$$W_m = \frac{1}{2} \int_V \vec{A}_p \cdot \vec{J} dV. \quad (12)$$

Using the Green's function in the three dimensional case and the solution of the vector potential

$$G(\vec{r}, \vec{r}') = \frac{\mu_0}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|}; \quad (13)$$

$$\vec{A}_p(\vec{r}) = \int_{sc} \vec{J}(\vec{r}') G(\vec{r}, \vec{r}') d\vec{r}', \quad (14)$$

the magnetic energy is expressed in terms of the current density:

$$W_m = \frac{1}{2} \int_{sc} \int_{sc} \vec{J}(\vec{r}) \cdot \vec{J}(\vec{r}') G(\vec{r}, \vec{r}') dV dV'. \quad (15)$$

This equation contains an integration over the superconducting volume (sc).

The energy of an external homogeneous field is given by:

$$W_f = \frac{1}{2} \int_{sc} \vec{A}_{pf} \cdot \vec{J} dV \quad (16)$$

where $\vec{A}_{pf}$ is the vector potential of the external field. The superconductor is discretized in small cuboids with a constant current density. For N small cuboids the total energy can be obtained using:

$$\begin{aligned}W &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \vec{J}_i(\vec{r}_i) \cdot \vec{J}_j(\vec{r}_j') \int_{sc} \int_{sc} G(\vec{r}_i, \vec{r}_j') dV_i dV_j \\ &+ \frac{\mu_0}{2} \sum_{i=1}^N \lambda_i^2 \vec{J}_i^2 V_i + \frac{1}{2} \sum_{i=1}^N \vec{A}_{pfi} \cdot \vec{J}_i V_i.\end{aligned} \quad (17)$$

The variation of the energy over the vector of the current densities leads to a linear system of equations. The continuity of current flow $\int_A \vec{J} \cdot d\vec{A} = 0$ is fulfilled using circulating currents on a mesh, as shown in Fig. 1a. The superposition of these currents leads to line currents in small cuboids on a different mesh for each direction (s. Fig. 1b,c). In order to solve the system of equations, sources are necessary. These sources are external fields or circulating currents around loops. For the solution of

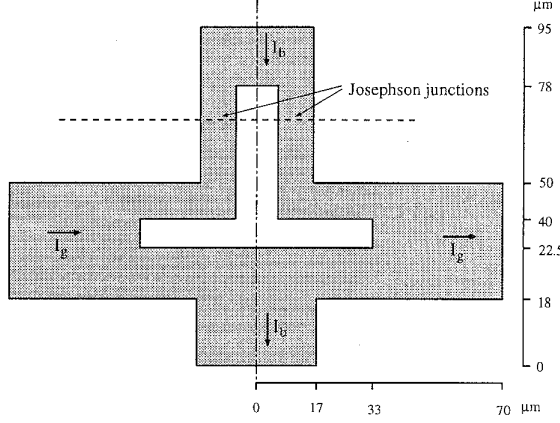


Fig. 2. Layout of the dc-SQUID

the field problem, it does not matter whether the source currents flow on the inner or the outer edge of the loop.

The set of currents in the small cuboids is united in the vector $\underline{I}_l$. This vector $\underline{I}_l$ can be expressed with the vector of the circulating currents $\underline{I}_c$, with the vector of the source currents $\underline{I}_s$ and an incidence matrix $\underline{Z}$ containing elements with the values -1, 0 and 1:

$$\underline{I}_l = \underline{Z} \underline{I}_c + \underline{I}_s. \quad (18)$$

The i -th line of the linear system of equations has been obtained by using the first derivatives of the energy (17) $\frac{\partial W}{\partial I_{li}} = 0$. The system of equations is given by:

$$\underline{M} \underline{I}_l + \underline{F} = 0, \quad (19)$$

where the vector $\underline{F}$ results from the external magnetic field. Using (18) and (19) the following system of equations is then obtained:

$$\underline{M}(\underline{Z} \underline{I}_c + \underline{I}_s) + \underline{F} = 0. \quad (20)$$

By multiplying (20) with the matrix $\underline{Z}^T$ the number of equations can be reduced to the number of circulating currents:

$$\underline{Z}^T \underline{M} \underline{Z} \underline{I}_c = -\underline{Z}^T \underline{M} \underline{I}_s - \underline{Z}^T \underline{F}. \quad (21)$$

The right hand side of the system of equations depends on the sources used, and the matrix $\underline{Z}^T \underline{M} \underline{Z}$ is determined by discretization and integration of the Green's function. An approximation of the integral of Green's function has been obtained by carrying out the integration over one single cuboid:

$$\begin{aligned} & \frac{\mu_0}{4\pi} \int_V \int_{V'} \frac{1}{r} dx dy dz dx' dy' dz' \\ &= \frac{\mu_0}{4\pi} \left[-\frac{Z^2}{2} \operatorname{atan} \left(\frac{XY}{Zr} \right) - \frac{Y^2}{2} \operatorname{atan} \left(\frac{XZ}{Yr} \right) \right. \\ & \quad - \frac{X^2}{2} \operatorname{atan} \left(\frac{YZ}{Xr} \right) + YZ \ln(X+r) \\ & \quad + \left. XZ \ln(Y+r) + XY \ln(Z+r) \right] V' \left| \begin{matrix} x_1 & y_1 & z_1 \\ x_0 & y_0 & z_0 \end{matrix} \right|, \end{aligned} \quad (22)$$

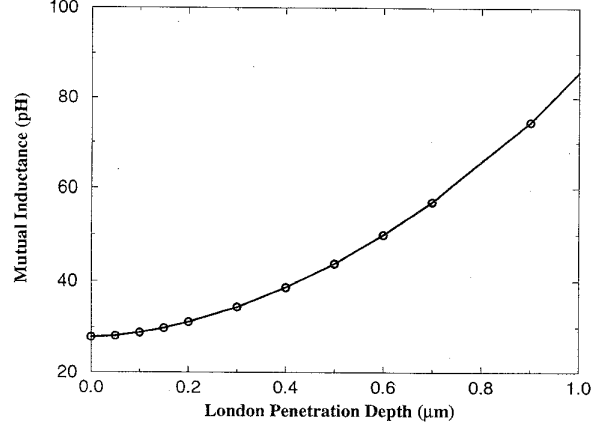


Fig. 3. Calculated inductance as a function of the London penetration depth.

where the following abbreviations are used:

$$\begin{aligned} X &= x - x', \quad Y = y - y', \quad Z = z - z', \\ r &= \sqrt{X^2 + Y^2 + Z^2}. \end{aligned} \quad (23)$$

The numerical computations show, that if the distance between the cuboids is five times larger than the size of the cross section dimension of the cuboids, (22) can be approximated asymptotically by:

$$\frac{\mu_0}{4\pi} \int_V \int_{V'} \frac{1}{r} dx dy dz dx' dy' dz' = \frac{\mu_0}{4\pi} \frac{1}{r_{ij}} V V'. \quad (24)$$

The inductances are obtained by $L = 2W/I_s^2$ and the mutual inductances by $M_{ij} = 2(W_{ii} + W_{jj} - W_{ij})/I_{sij}^2$, where W is calculated with (17). The value of the magnetic field at a point can be calculated using the vector potential $\vec{A}_p$ in (22), (14) and (8). The algorithm described has been realized in the computer program SCIM.

III. RESULTS OF THE CALCULATION

The results of the program SCIM have been compared to measured data of a high T_c dc-SQUID. The layout of the dc-SQUID is shown in Fig. 2. In this figure, the bias current of the SQUID I_b and the control current I_g are marked. The SQUID consisted of a single superconducting $\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$ layer with a thickness of 120 nm. The Josephson junctions were prepared as step edge junctions. The current I_g was used in order to couple flux into the SQUID-loop. The value of the current difference ΔI_g required to change the flux in the SQUID by one flux quantum $\Phi_0 \simeq 2.07$ Vs was measured as a function of temperature. The mutual inductance M was then calculated using: $M = \Phi_0 / \Delta I_g$. The mutual inductance changes with the temperature, since the London penetration depth in $\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$ was temperature dependent with the relation [10]:

$$\lambda(T) = \lambda(T = 0\text{K}) \left(1 - (T/T_c)^2 \right)^{-\frac{1}{2}}. \quad (25)$$

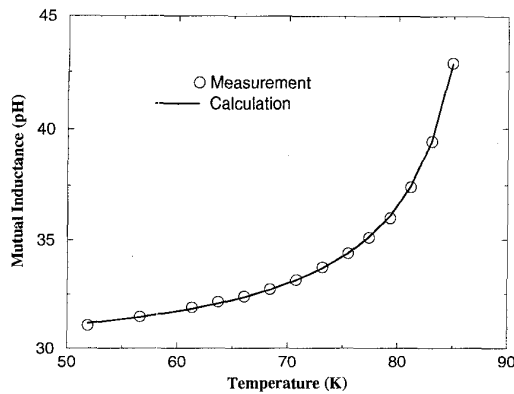


Fig. 4. Calculated and measured inductance as a function of the temperature

With this relationship and the numerical results of the program SCIM (see Fig. 3), the mutual inductances were obtained as a function of the temperature. Fig. 4 shows the comparison between the measured and the calculated mutual inductances from 52 K up to 84 K. The following parameters have been used: critical temperature of the superconductor $T_c = 90.3$ K and the penetration depth $\lambda(T = 0\text{K}) = 170$ nm.

The calculated current density distribution in the superconducting layer with a penetration depth $\lambda = 300$ nm is represented in Fig. 5 by arrows and a grey scale. Inside the loop the magnetic field distribution is represented by a contour plot. In this calculation, a circulating current around the hole has been used as source. For these calculations about 800 circulating currents have been used.

IV. CONCLUSIONS

With use of the presented algorithm and the program SCIM, the inductances and the field and current distribution in superconducting structures can be calculated. The results agree well with experimental results obtained from a high-temperature dc SQUID.

The important features of this program are: The discretization of the superconductor adapts automatically when elements with different sizes are used. Therefore, the solution becomes more independent of the initial discretization. Because the exact field solution for a cuboid with homogeneous current density is used, the field distribution is obtained accurately inside and outside of the superconductor. The direction of the current density is a result of the computation. The inductances of the structure are calculated via the energy and no additional discretization is necessary in order to integrate the magnetic flux. In addition to source currents, an external magnetic field can be used to investigate the field and current distribution under several conditions. The calculation time on workstations is in the range of a few seconds to some minutes. This program is a useful tool for the design of

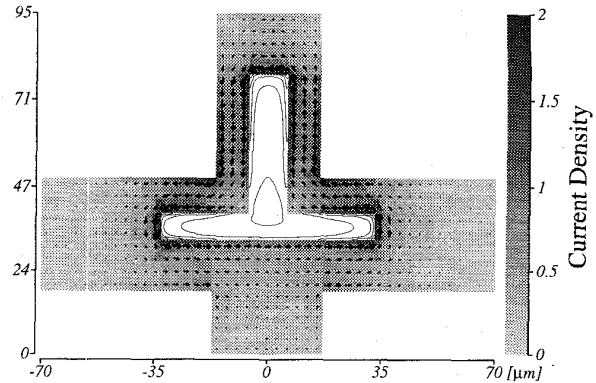


Fig. 5. Current density and magnetic field distribution in the structure.

high and low temperature superconducting circuits.

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