

COMPUTATIONAL METHOD OF THREE DIMENSIONAL EDDY CURRENT BY USING VOLUME INTEGRAL EQUATION METHOD

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Abstract - A three dimensional computational method for eddy current has been developed. Its characteristics are as follows: (1) An IEM approach based on the integral expression of magnetic vector potential is applied to calculate eddy current and magnetization. (2) A fractional step method is used for integration in the time direction to maintain current density conservation. (3) Edge and constant elements are used for the calculation of eddy current and magnetization, respectively, to discretize the equation in the spatial direction. (4) The method can be applied to a structure having holes where element division is not required.

The method has been applied to the eddy current problem of a thin square plate and its suitability was confirmed.

INTRODUCTION

In order to analyze eddy current, several calculation methods have been proposed, such as the use of magnetic vector potential [1] [2], current vector potential [3] or physical values (for example, magnetic flux density [4]). The current vector potential is only defined in the regions where current conducts, but a disadvantage is that the method also needs element division in any holes which are located in a structure. A method which uses physical values as unknowns directly might give good numerical accuracy. But, a large number of unknowns would be needed and element division in holes would also be required. The FEM [1] [3] (finite element method) and BEM [2] [4] (boundary element method) can be used for the numerical procedure. The FEM requires many element divisions in free space. By contrast, the BEM does not require element divisions in free space except on the boundary of a structure, but it is difficult to apply to non-linear problems, since the physical property is assumed to be uniform in the calculation domain. Another approach, the IEM [5] (integral equation method) does not need to divide free space into many elements except in a structure. The IEM can be applied to non-linear problems, because no assumption is made in the calculation domain. Therefore we have adopted it to analyze eddy current problems and the integral expression of magnetic vector potential has been chosen for the formulation.

Conventionally, to satisfy conservation of eddy current, a large linear equation system which involves both magnetic vector and electric scalar potentials as unknowns has been solved directly. To reduce the matrix size, a penalty function method has been used instead of solving the large linear equation system. However, this approach has a disadvantage that solutions depend on the penalty parameter. Therefore, we have applied a fractional step method to this problem.

That is, our time integration scheme to calculate eddy current is composed of two steps. First is solving step of Faraday's law by using volume integral equation and second is correcting step of obtained the eddy current to satisfy the conservation law.

BASIC EQUATIONS

The basic equations for the eddy current are described by Maxwell's formulae in which the displacement current is neglected,

$$\text{rot } \mathbf{H} = \mathbf{i} \quad (1)$$

$$\text{div } \mathbf{B} = 0 \quad (2)$$

$$\text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3)$$

where $\mathbf{H}$ denotes magnetic intensity; $\mathbf{B}$, magnetic flux density; $\mathbf{i}$, current density; and $\mathbf{E}$, electric stress. Constitutive equations for magnetic substances and electric conductors are as follows,

$$\mathbf{B} = \mu \mu_0 \mathbf{H} = \mu_0 (\mathbf{H} + \mathbf{M}) \quad (4)$$

$$\mathbf{i} = \sigma \mathbf{E} \quad (5)$$

where μ indicates permeability of the material; μ_0 , permeability of free space; $\mathbf{M}$, magnetization vector; and σ , conductivity of the material.

The following boundary conditions should be satisfied on the surface where two regions l and m make contact with each other,

$$\mathbf{B}_n^* + \mathbf{B}_n^l = 0 \quad (6)$$

$$\mathbf{H}_t^* = \mathbf{H}_t^l \quad (7)$$

where, subscripts n and t denote outward normal and tangential directions of the surface, respectively.

FORMULATION BY INTEGRAL EQUATION

Current density can be decomposed into eddy current $\mathbf{i}_e$ and given current density $\mathbf{i}_0$.

$$\mathbf{i} = \mathbf{i}_e + \mathbf{i}_0 \quad (8)$$

Magnetic intensity and vector potential can also be decomposed into three components,

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_e + \mathbf{H}_m \quad (9)$$

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_e + \mathbf{A}_m \quad (10)$$

where $\mathbf{H}_0, \mathbf{H}_e, \mathbf{H}_m$ and $\mathbf{A}_0, \mathbf{A}_e, \mathbf{A}_m$ are magnetic intensity and magnetic vector potential with respect to a given current, eddy current and existence of magnetic substance, respectively.

Each magnetic intensity and vector potential are described by Bio-Savart's law,

$$\mathbf{A}_0 = \mu_0 \int \mathbf{i}_0 \psi d\Omega \quad (11)$$

$$\mathbf{A}_e = \mu_0 \int \mathbf{i}_e \psi d\Omega \quad (12)$$

$$\mathbf{A}_m = \mu_0 \left\{ \int \mathbf{M} \times \mathbf{n} \psi d\Gamma + \int \text{rot } \mathbf{M} \psi d\Omega \right\} \quad (13)$$

$$\mathbf{H}_0 = \int \mathbf{i}_0 \times \text{grad} \psi \, d\Omega \quad (14)$$

$$\mathbf{H}_e = \int \mathbf{i}_e \times \text{grad} \psi \, d\Omega \quad (15)$$

$$\mathbf{H}_m = \left\{ \int \mathbf{M} \cdot \mathbf{n} \text{grad} \psi \, d\Gamma - \int \text{div} \mathbf{M} \text{grad} \psi \, d\Omega \right\} \quad (16)$$

where ψ is Green's function in free space.

$$\psi = \frac{1}{4\pi |\mathbf{r}_q - \mathbf{r}_p|} \quad (17)$$

In the above equation, $\mathbf{r}_p$ indicates the location vector of the observation point and $\mathbf{r}_q$ is that of the integration point in materials.

To determine the magnetization in Eq. (16) the magnetic moment method is adopted, it can satisfy the boundary conditions of Eqs. (6) and (7) automatically.

$$\mathbf{M} + (1-\mu) \left\{ \int \mathbf{M} \cdot \mathbf{n} \text{grad} \psi \, d\Gamma - \int \text{div} \mathbf{M} \text{grad} \psi \, d\Omega \right\} = (\mu-1) (\mathbf{H}_0 + \mathbf{H}_e) \quad (18)$$

Equation (3) is rewritten by using the total vector potential and electric scalar potential Φ as follows,

$$\rho \mathbf{i}_e = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad} \Phi \quad (19)$$

where ρ is resistivity of the conductors.

Substituting Eq. (10) into Eq. (14) and considering the vector potential expressed by Eqs. (11)-(13), the following equation can be obtained.

$$\rho \mathbf{i}_e = -\mu_0 \frac{\partial}{\partial t} \left(\int \mathbf{i}_0 \psi \, d\Omega + \int \mathbf{i}_e \psi \, d\Omega + \int \mathbf{M} \times \mathbf{n} \, d\Gamma - \int \text{rot} \mathbf{M} \psi \, d\Omega \right) - \text{grad} \Phi \quad (20)$$

Equation (18) can be rewritten by considering Eqs. (14) and (15).

$$\mathbf{M} + (1-\mu) \left\{ \int \mathbf{M} \cdot \mathbf{n} \text{grad} \psi \, d\Gamma - \int \text{div} \mathbf{M} \text{grad} \psi \, d\Omega \right\} = (\mu-1) \left(\int \mathbf{i}_0 \times \text{grad} \psi \, d\Omega + \int \mathbf{i}_e \times \text{grad} \psi \, d\Omega \right) \quad (21)$$

Equations (20) and (21) are an integral equation system in which unknowns are eddy current density and magnetization vector.

By discretizing these equations in time and space directions, distributions of the eddy current and magnetization can be obtained.

The present method based on Eqs. (11)-(13) can satisfy Coulomb gauge automatically, because divergence of these equations is zero, due to conservation of the current density.

DISCRETIZATION

(1) Discretization in time direction

The fractional step method has been used in the field of numerical fluid mechanics [6] and its effectiveness has been confirmed to satisfy conservation of flow rate. Conventionally, it has not been used for eddy current analysis. We adopt this method to satisfy conservation of current density. Eddy current between one time increment can be expressed by using the parameter ω ,

$$\mathbf{i}_e = \omega \mathbf{i}_e^\tau + (1-\omega) \mathbf{i}_e^0 \quad (22)$$

where superscripts τ and 0 indicate the values at time τ and initial time, respectively. Equation (20) can be discretized considering Eq. (22).

$$\begin{aligned} \omega \tau \rho \mathbf{i}_e^\tau + \mu_0 \int \mathbf{i}_e^\tau \psi \, d\Omega = & \mu_0 \left\{ \int \mathbf{i}_e^0 \psi \, d\Omega - \int (\mathbf{i}_e^\tau - \mathbf{i}_e^0) \psi \, d\Omega - \int (\mathbf{M}^\tau - \mathbf{M}^0) \times \mathbf{n} \psi \, d\Gamma \right. \\ & \left. + \int \text{rot} (\mathbf{M}^\tau - \mathbf{M}^0) \psi \, d\Omega \right\} - (1-\omega) \tau \rho \mathbf{i}_e^0 - \\ & \tau \text{grad} (\omega \Phi^\tau + (1-\omega) \Phi^0) \end{aligned} \quad (23)$$

If ω equals unity, Eq. (23) becomes a full implicit time integration scheme, and if ω is 0.5, it becomes the Crank-Nicholson scheme. Solution of Eq. (23) might not satisfy conservation of current density. Then, we introduce the prediction factor $\mathbf{i}_e^*$ and correction factor $\delta \mathbf{i}$ and express $\mathbf{i}_e^\tau$ as follows.

$$\mathbf{i}_e^\tau = \mathbf{i}_e^* + \delta \mathbf{i} \quad (24)$$

Equation (23) can be decomposed into fractional steps as follows,

$$\begin{aligned} \omega \tau \rho \mathbf{i}_e^* + \mu_0 \int \mathbf{i}_e^* \psi \, d\Omega = & \mu_0 \left\{ \int \mathbf{i}_e^0 \psi \, d\Omega - \int (\mathbf{i}_e^\tau - \mathbf{i}_e^0) \psi \, d\Omega - \int (\mathbf{M}^\tau - \mathbf{M}^0) \times \mathbf{n} \psi \, d\Gamma \right. \\ & \left. + \int \text{rot} (\mathbf{M}^\tau - \mathbf{M}^0) \psi \, d\Omega \right\} - (1-\omega) \tau \rho \mathbf{i}_e^0 - \\ & \tau \text{grad} \Phi^0 \end{aligned} \quad (25)$$

$$\rho \delta \mathbf{i} = -\text{grad} \delta \Phi \quad (26)$$

where $\delta \Phi$ is the increment of electric scalar potential between one time step.

$$\Phi^\tau = \Phi^0 + \delta \Phi \quad (27)$$

In order to satisfy the conservation of current density at time τ , divergence of the current must be zero, that is:

$$\text{div} \mathbf{i}_e^\tau = \text{div} \mathbf{i}_e^* + \text{div} \delta \mathbf{i} = 0 \quad (28)$$

Substituting Eq. (26) into Eq. (28), the following equation can be obtained.

$$\text{div} \sigma \text{grad} \delta \Phi = \text{div} \mathbf{i}_e^* \quad (29)$$

After solving Eq. (25), $\delta \Phi$ can be determined by Eq. (29) and $\delta \mathbf{i}$ is obtained by substituting $\delta \Phi$ into Eq. (26). In this manner, new eddy current density at time equals τ can be calculated.

(2) Discretization in spatial direction

To calculate magnetization and eddy current density distributions, the materials are divided into many elements. Equations (25), (26) and (29) have been discretized by the Galerkin method and Eq. (21), by the collocation method. To calculate vector potential and current density, a linear edge element is used,

$$\mathbf{A} = \sum_j N_j^* \mathbf{e}_j A_j \quad (30)$$

$$\mathbf{i} = \sum_j N_j^* \mathbf{e}_j i_j \quad (31)$$

where N_j^* is a shape function for edge element, A_j and i_j indicate vector potential and current density on the j -th element edge, and $\mathbf{e}_j$ is the unit vector of the edge. Electric scalar potential and location vector $\mathbf{r}$ in an element are interpolated by the normal linear shape function N_j ,

$$\Phi = \sum_j N_j \Phi_j \quad (32)$$

$$\mathbf{r} = \sum_j N_j \mathbf{r}_j \quad (33)$$

where Φ_j and $\mathbf{r}_j$ indicate electric scalar potential and location vector at the j -th nodal point in an element. For calculation of magnetization, a constant element has been selected. The evaluation point of the magnetization is the center of gravity.

Under the above preparation, Eq. (21) can be expressed in a matrix form as follows,

$$\{[E] + (1-\mu)[K]\}\{\mathbf{M}\} = (\mu-1)\{[U_o]\{i_o\} + [U_e]\{i_e\}\} \quad (34)$$

where $[E]$ indicates a unit matrix. Matrices $[K]$, $[U_o]$ and $[U_e]$ are

$$[K] = [\sum_m \sum_{\Delta \Gamma \in \Delta \Omega_m} \int_{\Delta \Gamma} n \text{grad} \psi d\Gamma]$$

$$[U_o] = [\sum_m \int_{\Delta \Omega_m} N_j^* \mathbf{e}_j \times \text{grad} \psi d\Omega_o]$$

$$[U_e] = [\sum_m \int_{\Delta \Omega_m} N_j^* \mathbf{e}_j \times \text{grad} \psi d\Omega_e]$$

Matrices $[U_o]$ and $[U_e]$ are similar to each other, but, the integration region of the given and eddy current differ from each other, so, different forms are defined for each matrix. In Eq. (34), the volumetric integral disappears, because magnetization $\mathbf{M}$ is assumed to be constant in an element and the differential of $\mathbf{M}$ becomes zero.

The discretized form of Eq. (25) is described by a matrix expression.

$$\begin{aligned} [S]\{\omega \tau \rho [E] + \mu_o [P_e]\}\{i_e^*\} &= \mu_o [S]\{[P_e]\{i_e^0\} - [P_o]\{i_o^r - i_o^0\} - \\ [P_m]\{\mathbf{M}^r - \mathbf{M}^0\}\} - (1-\omega) \tau \rho [S]\{i_e^0\} - \\ \tau [G]\{\Phi^0\} \end{aligned} \quad (35)$$

Matrices in Eq. (35) are as follows.

$$[S] = [\sum_m \int_{\Delta \Omega_m} N_j^* N_i^* \mathbf{e}_j \cdot \mathbf{e}_i d\Omega]$$

$$[P_o] = [\sum_m \int_{\Delta \Omega_m} N_j^* \mathbf{e}_j \cdot \mathbf{e}_p \psi d\Omega_o]$$

$$[P_e] = [\sum_m \int_{\Delta \Omega_m} N_j^* \mathbf{e}_j \cdot \mathbf{e}_p \psi d\Omega_e]$$

$$[P_m] = [\sum_m \sum_{\Delta \Gamma \in \Delta \Omega_m} \int_{\Delta \Gamma} \mathbf{e}_q \times \mathbf{n} \psi d\Gamma]$$

$$[G] = [\sum_m \int_{\Delta \Omega_m} N_i^* \text{grad} N_j \cdot \mathbf{e}_i d\Omega]$$

Equations (27) and (29) become

$$[S]\{\delta i\} = -\sigma [G]\{\delta \Phi\} \quad (36)$$

$$[D]\{\delta \Phi\} = [G]^T \{i_e^*\} \quad (37)$$

where

$$[D] = [\sum_m \int_{\Delta \Omega_m} \sigma \text{grad} N_j \text{grad} N_i d\Omega]$$

and the superscript T means a transposed matrix.

Substituting Eq. (34) into Eq. (35), the

following final form is obtained.

$$\begin{aligned} [S]\{\omega \tau \rho [E] + \mu_o [P_e] + \mu_o [P_m][T_e]\}\{i_e^*\} \\ = \mu_o [S]\{([P_e] + [P_m][T_e])\{i_e^0\} - \\ ([P_o] + [P_m][T_o])\{i_o^r - i_o^0\} - \\ (1-\omega) \tau \rho [S]\{i_e^0\} - \tau [G]\{\Phi^0\}\} \end{aligned} \quad (38)$$

where

$$[T_e] = \{[E] + (1-\mu)[K]\}^{-1} (\mu-1)[U_e]$$

$$[T_o] = \{[E] + (1-\mu)[K]\}^{-1} (\mu-1)[U_o]$$

The calculation procedures of eddy current are as follows.

- ① The prediction factor is calculated by Eq. (38).
- ② $\delta \Phi$ is obtained by substituting the results of ① into Eq. (37).
- ③ The correction factor is calculated by using Eq. (36).
- ④ Adding the correction factor to the prediction factor results in eddy current at time equals τ .
- ⑤ Calculation of the time dependent eddy current can be performed by repeating procedures ① to ④.

VERIFICATION OF THE METHOD

In order to verify the method, it was applied to the square thin plate problem as illustrated in Fig. 1. The uniform magnetic field was applied to the plate and the increasing rate of magnetic flux density was 15T/s. The thickness of the square was 2 mm and its side length was 20mm. Conductivity of the material was 5×10^7 S/m and permeability was 1.0. The analytical region was selected considering geometrical symmetry.

Calculation results are shown in Figs. 2 and 3. Fig. 2 indicates current distribution on the plate at 12ms from the initiation of the magnetic field rising and Fig. 3 is the magnetic flux density distribution at the same time. The maximum eddy current density is about 1.7×10^7 A/m² on the edge of the plate and the minimum value is zero at the center of the plate. Magnetic flux density also has a maximum value on the edge, and a minimum at the center. This is because the eddy current is induced to maintain a constant magnetic flux density. In this way, the method gives reasonable results physically.

Figure 4 compares the calculation results with an exact solution of the plate when the eddy current reaches the terminal value under the calculation conditions. In the figure, the Y-direction component of the eddy current density is plotted along the X-axis. As shown in the figure the calculation results agree with the exact solution within 1%.

CONCLUSION

In order to calculate three dimensional eddy current, a computational method based on IEM has been developed. The characteristics of the method are as follows:

- (1) An integral form of the magnetic vector potential and magnetic moment method are applied to calculate eddy current and magnetization, respectively. The integral form satisfies the Coulomb gauge with respect to magnetic vector potential automatically.
- (2) The fractional step method is used for integration in the time direction to maintain current density conservation. This reduces

matrix size for the linear equation system.

(3) To discretize the integral equation in the spatial direction, edge and constant elements are used for the calculation of eddy current and magnetization, respectively.

(4) The method can be applied to a structure having holes where element division is not required.

To verify the present method, it has been applied to the eddy current problem of a thin square plate. The calculation results agreed with the exact solution within 1% and its suitability was confirmed.

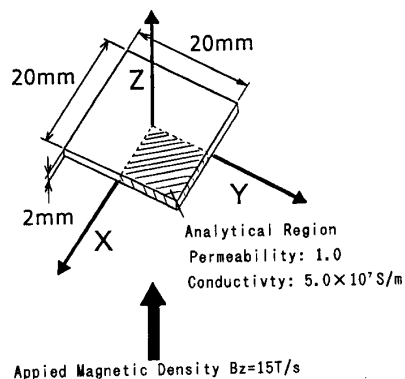


Fig.1 Calculation Conditions of Thin Square Plate

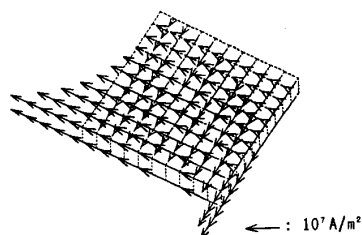


Fig.2 Current Density Distribution
(at 12ms)

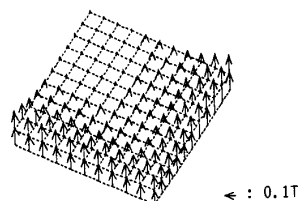


Fig.3 Magnetic Flux Density Distribution
(at 12ms)

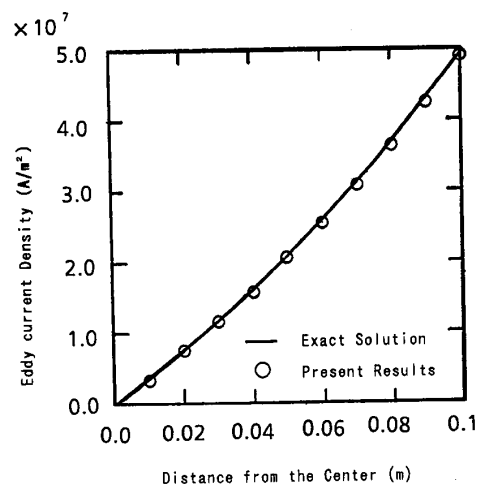


Fig.4 Y-Component of Terminal Eddy Current Density
along the X-Axis

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