

Comparison of MoM/SIE, MoM/VIE and FEM Based on Topological Analysis of Two Canonical Problems

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1. Introduction

Analysis of composite metallic and dielectrics structures placed in time-harmonic electromagnetic field is usually based on the MoM (Method of Moments) [1], or FEM (Finite Element Method) [2]. Particularly, if MoM is applied to SIE (Surface Integral Equation) [3,4], the method is termed as MoM/SIE, and if MoM is applied to VIE (Volume Integral Equation) [5], the method is termed as MoM/VIE. Besides that, all these methods can be mutually combined giving different types of hybrid methods.

It is well known that none of these methods is the most efficient in all cases. Very often it is considered that:

- A. the MoM is more convenient for open problems than the FEM, and vice versa, the FEM is more convenient for closed problems than the MoM;
- B. the MoM/SIE is well suited for wire and plate structures, while the MoM/VIE and the FEM are well suited for dielectric structures;
- C. the MoM matrices are always full, while the FEM matrices are sparse;
- D. the MoM is more efficient in the case of small problems, while FEM is more efficient in the case of large problems.

Actually, these methods are compared very rarely. According to the best knowledge of the authors there is no any quantitative rule, that can be used for making decision which method is optimal for solution of a specific problem.

The first goal of the paper is to show that very often above mentioned rules (A, B, C and D) are not correct. The second goal of the paper is to define some quantitative rules that can be used for comparison of the methods. These goals are achieved by topological analysis of two canonical problems.

2. Two canonical problems

Let us consider a cubical body of side a , consisting of $m_x \times m_y \times m_z$ smaller cubes, made of different linear isotropic and homogenous dielectrics. If the body is completely covered by PEC (Perfect Electric Conductor), the problem is closed (1st canonical problem), otherwise the problem is opened (2nd canonical problem). Note that in the case of the closed problem the MoM/VIE (applied to dielectric) must be combined with the MoM/SIE (applied to PEC). Similarly, in the case of the open problem it is supposed that the weak FEM formulation is combined with MoM/SIE. In what follows these actually hybrid methods will be termed as MoM/VIE and FEM.

In the case when analysis is performed by the MoM/SIE each square patch of edge $b=a/m_d$ is subdivided into $m_p \times m_p$ patches. In the case when analysis is performed by the MoM/VIE or the FEM each cube of edge $b=a/m_d$ is subdivided into $m_p \times m_p \times m_p$ smaller cubes. In both cases m_p depends on the electrical length of edge b and order of approximation used along edge b .

For the beginning, let us suppose that the lowest order expansions for currents over square patches and fields over cubes (surface and volume rooftop basis functions) are applied. Optimal basis functions for the MoM/SIE satisfy continuity of current component normal to patch sides. Optimal basis functions for the FEM satisfy continuity of the field component tangential to cube sides. The same basis functions can be used for the MoM/VIE. In all these cases basis functions are associated to edges (i.e. patch edges and cube edges). It can be shown that number of unknowns per inner edge (shared by few patches or cubes) is 2 for MoM/SIE (1 for electric currents and 1 for magnetic currents) and 1 for other two methods. If the problem is closed the number of unknowns per outer edge is 1 for MoM/SIE and MoM/VIE, and 0 for FEM. If the problem is opened the number of unknowns per outer edge is 2 for MoM/SIE and FEM, and 1 for MoM/VIE.

In the general case, let the expansion of order n_p be used over the square patch (cube) of edge $c=b/m_p=a/m_p m_d$. Let higher order expansion be constructed in such a manner that for $n_p=1$ rooftop basis functions are obtained. It can be shown that number of unknowns used by n_p^{th} order expansions is the same as in the case when the square patch (cube) is divided into $n_p \times n_p$ ($n_p \times n_p \times n_p$) square patches (cubes) and 1st order expansions are applied.

3. Number of unknowns

Total number of edges, M_V , and number of outer edges, M_S , occurring in the geometrical model used in MoM/SIE analysis can be expressed as

$$M_V = 6m^2 m_d + 6m^2 - 3mm_d^2 + 3m \quad M_S = 12m^2 \quad (1a,b)$$

where $m=m_p m_d$. If $n_p=1$ the corresponding number of unknowns is

$$N_{\text{open}}^{(\text{SIE})} = 2M_V \quad N_{\text{close}}^{(\text{SIE})} = 2M_V - M_S \quad (2a,b)$$

Finally, by expressing M_V and M_S in eqns. (2a,b) according to eqns. (1a,b) and replacing m_p by $n_p m_p$, the number of unknowns for arbitrary n_p is obtained as

$$N_{\text{open}}^{(\text{SIE})} = 12n^2 m_d + 12n^2 - 6nm_d^2 + 6n \quad N_{\text{close}}^{(\text{SIE})} = 12n^2 m_d - 6nm_d^2 + 6n \quad (3a,b)$$

where $n=n_p m_p m_d$.

If $n_p=1$ and $m_p=1$, the number of unknowns used in FEM and MoM/VIE analysis can be written as

$$N_{\text{open}}^{(\text{FEM})} = M_V + M_S \quad N_{\text{close}}^{(\text{FEM})} = M_V - M_S \quad (4a,b)$$

$$N_{\text{open}}^{(\text{VIE})} = N_{\text{close}}^{(\text{VIE})} = M_V \quad (4c)$$

By expressing M_V and M_S in eqns. (4a,b,c) according to eqns. (1a,b) and having in mind that $n=m=m_d$, the number of unknowns is obtained as

$$N_{\text{open}}^{(\text{FEM})} = 3n^3 + 18n^2 + 3n \quad N_{\text{close}}^{(\text{FEM})} = 3n^3 - 6n^2 + 3n \quad (5a,b)$$

$$N_{\text{open}}^{(\text{VIE})} = N_{\text{close}}^{(\text{VIE})} = 3n^3 + 6n^2 + 3n \quad (5c)$$

The same expressions are valid for arbitrary n_p and m_p .

By comparing eqns. (3a,b) and (5a,b,c) it is found that the number of unknowns used by MoM/SIE is less than the number of unknowns used by other two methods under the following conditions:

a) $N_{\text{open}}^{(\text{MoM})} < N_{\text{open}}^{(\text{VIE})}$ and $N_{\text{close}}^{(\text{MoM})} < N_{\text{close}}^{(\text{FEM})}$ if

$$n_d > \left(2 + \sqrt{2}\right) \left(1 + \frac{1}{\sqrt{2}m_d}\right) \approx 3.4 + \frac{2.4}{m_d} \quad (6a)$$

b) $N_{\text{open}}^{(\text{MoM})} < N_{\text{open}}^{(\text{FEM})}$ and $N_{\text{close}}^{(\text{MoM})} < N_{\text{close}}^{(\text{VIE})}$ if

$$n_d > \left(2 + \sqrt{2}\right) \left(1 - \frac{1}{\sqrt{2}m_d}\right) \approx 3.4 - \frac{2.4}{m_d} \quad (6b)$$

where $n_d = n/m_d$ is the number of unknowns per edge b .

Finally, it can be concluded that:

- 1) MoM/VIE requires less number of unknowns than FEM for open problems,
- 2) FEM requires less number of unknowns than MoM/VIE for closed problems
- 3) MoM/SIE requires less number of unknowns than MoM/VIE and FEM if $n_d > 5$ for each m_d , i.e. if $n_d > 3$ for large m_d ,

However, efficiency of the solution depends not only on the number of unknowns, but on the total number of operations.

4. Number of operations

It is very difficult to compare the total number of operations in the general case. On the other side, it is much more important to make such comparison in the case of complex and electrically large problems. For such problems the main part of the number of operations is due to the solution of system of linear equations. In the case of full and sparse matrices, the number of basic operations, K , can be estimated as

$$K_{\text{full}} = \frac{1}{3} N^3 \quad K_{\text{sparse}} = NL^2 \quad (7)$$

where N is the number of unknowns, L is the half-bandwidth of the sparse matrix, and a basic operation consists of one summation and one multiplication.

In the case of full matrices, the efficiency of the method is directly related to the number of unknowns, as is discussed in the previous section. Note that for complex and electrically large problems the total number of unknowns can be written in an approximate form as

$$N^{(\text{SIE})} = 12n^2m_d - 6nm_d^2 \quad N^{(\text{FEM})} = N^{(\text{VIE})} = 3n^3 \quad (8)$$

In the case of MoM/VIE matrices are always full. However, in the case not only of FEM, but also of MoM/SIE unknown coefficients can be numerated in such a manner that matrices are sparse. In the case of closed problems half-bandwidths for the FEM and the MoM/SIE are

$$L_{\text{close}}^{(\text{FEM})} = \frac{N}{m_p m_d} \quad L_{\text{close}}^{(\text{SIE})} = 2 \frac{N}{m_d} \quad (9)$$

It is obvious that K for sparse matrices is essentially lower than K for full matrices only if $L = N/\sqrt{3}$. In the case of large problems $m = m_p m_d$ is always much greater than unity. It means that in the case of FEM the matrices should be always treated as sparse. However, in the case of MoM/SIE the matrices should be treated as sparse only if $m_d \geq 4$. Finally, note that above half-bandwidths can be mutually related as

$$L^{(\text{SIE})} = 2m_p L^{(\text{FEM})} \quad (10)$$

Having in mind equations (7)-(10) the total number of unknowns used by MoM/SIE can be related to the total number of unknowns used by FEM as

$$K^{(\text{SIE})} = K^{(\text{FEM})} \frac{256}{n_p^2 n_d} \left(1 - \frac{1}{2n_d}\right)^3 \quad (11)$$

The same relation is approximately valid in the case of open structures. It is seen from the above formula that for small enough n_d FEM is more efficient than MoM/SIE, while for large enough n_d , MoM/SIE is more efficient than FEM. Particularly, for large n_d it is obtained that

$$K^{(\text{SIE})} < K^{(\text{FEM})} \quad \text{if} \quad n_d \geq \max\left(n_p, \frac{256}{n_p^2}\right) \quad (12)$$

It is seen from eqns. (11) and (12) that MoM/SIE is more efficient than FEM, if higher order expansions, such that $n_p > 5$, are applied. (Note that for $m_d < 4$ the same conclusions can be valid even for $n_p > 2$.) However, such expansions should be used only if homogenous parts of bodies are electrically large enough. Usually orders $n_p > 5$ are used for dimensions greater than 1 wavelength.

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