

On the Choice of Optimal Basis Functions for MoM/SIE, MoM/VIE, FEM and Hybrid Methods

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1. Introduction

Analysis of composite metallic and dielectrics structures placed in time-harmonic electromagnetic field is usually based on the MoM (Method of Moments) [1], or FEM (Finite Element Method) [2]. Particularly, if MoM is applied to SIE (Surface Integral Equation), the method is termed as MoM/SIE, and if MoM is applied to VIE (Volume Integral Equation) the method is termed as MoM/VIE. Besides that, all these methods can be mutually combined giving different types of hybrid methods. Efficiency of all these methods depends a lot on the choice of basis functions. A variety of basis functions is used by different authors. Sometimes it seems that basis functions written in one notation and used by one method are completely different from and without any connection with basis functions written in another notation and used by another method. In addition, it seems that some of used basis functions are not optimally adopted.

The first goal of this paper is to show that practically all basis functions used in MoM/SIE, MoM/VIE and FEM belong to a few general classes of basis functions. The second goal is to establish the set of desired properties of basis functions used in each of the methods. Finally, the third goal is to compare different classes of basis functions starting from the set of desired properties, and choose the optimal classes of basis functions.

2. General classes of basis functions

First, all basis functions can be divided into two groups: surface basis functions and volume basis functions. In the general case surface basis function can be defined over curvilinear isoparametric surface, given by parametric equation $r=r(p,s)$, $0 \leq p \leq 1$, $0 \leq s \leq 1$. In the particular cases these surfaces degenerate into: curvilinear triangle, bilinear quadrilateral, and bilinear (flat) triangle. For example, a bilinear quadrilateral is defined by parametric equation

$$r(p,s) = r_c + r_p p + r_s s + r_{ps} ps \quad (1)$$

A flat triangle is defined by the same equation if $r_{ps}=0$. Practically all surface basis functions found in literature can be expressed in two general forms

$$f_s(p,s) = f(p,s) \frac{a_s}{S_0} \quad f_s(p,s) = f(p,s) \frac{n \times a_s}{S_0} \quad (2a,b)$$

where $f(p,s)$ is arbitrary known function, $a_p = dr(p,s)/dp$ and $a_s = dr(p,s)/ds$ are the so-called unitary vectors, n is unit vector normal to the surface and S_0 can be found in two variants

$$S_0 = |a_p \times a_s| \quad S_0 = |a_p \times a_s| \quad (3a,b)$$

First of basis functions (2) is intended to provide continuity of normal surface vector component at a junction of two patches. However, in the case of S_0 given by the first of expressions (3) the continuity can be satisfied only for rectangular and rhombic patches. Hence, in what follows only second of expressions (3) for S_0 will be used. In that case the second of basis functions (2) can provide continuity of tangential surface vector component at junction of two patches.

In the general case volume basis functions can be defined over curvilinear isoparametric hexahedrons, given by parametric equation $r=r(p,s)$, $0 \leq p \leq 1$, $0 \leq s \leq 1$, $0 \leq t \leq 1$. In the particular cases

these surfaces degenerate into: curvilinear tetrahedrons, trilinear hexahedrons, and trilinear tetrahedrons. For example, a trilinear hexahedron is defined by parametric equation

$$r(p, s, t) = r_p + r_{ps}p + r_{st}s + r_{pt}t + r_{pst}pst + r_{ps}p + r_{st}s + r_{pt}t + r_{pst}pst \quad (4)$$

A trilinear tetrahedron is defined by the same equation if $r_{ps}, r_{st}, r_{pt}, r_{pst} = 0$. Most often volume basis functions can be expressed in terms of two general forms

$$f_v(p, s, t) = f(p, s, t) \frac{a_i}{V_0} \quad f_v(p, s, t) = f(p, s, t) \frac{a_p \times a_s}{V_0} \quad (5a, b)$$

where $f(p, s, t)$ is arbitrary known function and V_0 can be written in two variants

$$V_0 = [a_p, a_s, a_t] \quad V_0 = (a_p \times a_s) \cdot a_t \quad (6a, b)$$

First of basis functions (5) is intended to provide continuity of normal volume vector component at a boundary surface of two domains. However, in the case of V_0 given by the first of expressions (6) the continuity can be satisfied only for some specific hexahedrons (e.g. parallelepipeds). Hence, in what follows only second of expressions (6) for V_0 will be used. In that case the second of basis functions (5) can provide continuity of tangential volume vector component at a boundary surface of two domains.

Generally speaking, usage of higher order basis functions enable more flexible and efficient analysis. However, it is desirable that even with the lowest order basis functions some requirements can be fulfilled. Hence, in what follows we shall concentrate on the lowest order basis functions.

3. Generalized quadrilateral and hexahedral rooftop basis functions

The lowest order basis functions that can provide the continuity of normal/tangential surface/volume vector component at the (junction of two patches)/(boundary surface of two domains) are obtained if unitary vectors belonging to numerators in eqns. (2) and (5) are multiplied by corresponding local coordinate (a_p by p , a_s by s and a_t by t). Particularly, in the case of bilinear surfaces and trilinear hexahedrons position vectors $\rho_p = pa_p$, $\rho_s = sa_s$ and $\rho_t = ta_t$ can be defined. Finally, having in mind the second of eqns. (3) and (6) the lowest order basis functions can be written in the form

$$f_s(p, s) = \frac{sa_s}{[a_p \times a_s]} \quad f_s(p, s) = \frac{\bar{\rho}_t}{[a_p \times a_s]} \quad (7a, b)$$

$$f_t(p, s) = \frac{n \times (sa_s)}{[a_p \times a_s]} \quad f_t(p, s) = \frac{n \times \bar{\rho}_t}{[a_p \times a_s]} \quad (8a, b)$$

$$f_v(p, s, t) = \frac{ta_t}{(a_p \times a_s) \cdot a_t} \quad f_v(p, s, t) = \frac{\bar{\rho}_t}{(a_p \times a_s) \cdot a_t} \quad (9a, b)$$

$$f_v(p, s, t) = \frac{(pa_p) \times (sa_s)}{(a_p \times a_s) \cdot a_t} \quad f_v(p, s, t) = \frac{\bar{\rho}_p \times \bar{\rho}_s}{(a_p \times a_s) \cdot a_t} \quad (10a, b)$$

where expression designated by (a) correspond to curvilinear isoparametric elements, while expression designated by (b) correspond to bilinear and trilinear elements. Note that basis functions given by eqns. (7b), (9b) and (10b) are used in Refs. 3-5.

4. Generalized triangular and tetrahedral rooftop basis functions

The lowest order basis functions in the case of triangular and tetrahedral isoparametric elements can be written in the form

$$N_{\text{open}}^{(\text{FEM})} = 3n^3 + 18n^2 + 3n \quad N_{\text{close}}^{(\text{FEM})} = 3n^3 - 6n^2 + 3n \quad (5a,b)$$

$$N_{\text{open}}^{(\text{VIE})} = N_{\text{close}}^{(\text{VIE})} = 3n^3 + 6n^2 + 3n \quad (5c)$$

The same expressions are valid for arbitrary n_p and m_p .

By comparing eqns. (3a,b) and (5a,b,c) it is found that the number of unknowns used by MoM/SIE is less than the number of unknowns used by other two methods under the following conditions:

a) $N_{\text{open}}^{(\text{MoM})} < N_{\text{open}}^{(\text{VIE})}$ and $N_{\text{close}}^{(\text{MoM})} < N_{\text{close}}^{(\text{FEM})}$ if

$$n_d > \left(2 + \sqrt{2}\right) \left(1 + \frac{1}{\sqrt{2}m_d}\right) \approx 3.4 + \frac{2.4}{m_d} \quad (6a)$$

b) $N_{\text{open}}^{(\text{MoM})} < N_{\text{open}}^{(\text{FEM})}$ and $N_{\text{close}}^{(\text{MoM})} < N_{\text{close}}^{(\text{VIE})}$ if

$$n_d > \left(2 + \sqrt{2}\right) \left(1 - \frac{1}{\sqrt{2}m_d}\right) \approx 3.4 - \frac{2.4}{m_d} \quad (6b)$$

where $n_d = n/m_d$ is the number of unknowns per edge b .

Finally, it can be concluded that:

- 1) MoM/VIE requires less number of unknowns than FEM for open problems,
- 2) FEM requires less number of unknowns than MoM/VIE for closed problems
- 3) MoM/SIE requires less number than MoM/VIE and FEM if $n_d > 5$ for each m_d , i.e. if $n_d > 3$ for large m_d .

However, efficiency of the solution depends not only on the number of unknowns, but on the total number of operations.

4. Number of operations

It is very difficult to compare the total number of operations in the general case. On the other side, it is much more important to make such comparison in the case of complex and electrically large problems. For such problems the main part of the number of operations is due to the solution of system of linear equations. In the case of full and sparse matrices, the number of basic operations, K , can be estimated as

$$K_{\text{full}} = \frac{1}{3} N^3 \quad K_{\text{sparse}} = NL^2 \quad (7)$$

where N is the number of unknowns, L is the half-bandwidth of the sparse matrix, and a basic operation consists of one summation and one multiplication.

In the case of full matrices, the efficiency of the method is directly related to the number of unknowns, as is discussed in the previous section. Note that for complex and electrically large problems the total number of unknowns can be written in an approximate form as

$$N^{(\text{SIE})} = 12n^2m_d - 6nm_d^2 \quad N^{(\text{FEM})} = N^{(\text{VIE})} = 3n^3 \quad (8)$$

Good results can be obtained even if some of desirable properties are not satisfied. But in these cases usually greater number of unknowns and longer analysis time is required. Having in mind Tables 2 and 3 it can be concluded that optimal rooftop basis functions are:

- given by eqns. (7b) or/and (10b) for QH (quadrilateral or/and. hexahedral) modeling, and
- given by eqns. (11b) or/and (14b) for TT (triangular or/and. tetrahedral) modeling

Table 2. Various types of rooftop basis functions and their properties.

basis function	1	2	3	4	5	6	basis functions	1	2	3	4	5	6
$\frac{\vec{p}_s}{[a_p \times a_s]}$	x		x	o		x	$\frac{\vec{p}_s}{2S}$	x		x	x		x
$\frac{n \times \vec{p}_s}{[a_p \times a_s]}$		x	o		o		$\frac{n \times \vec{p}_s}{2S}$		x	x		x	
$\frac{\vec{p}_t}{(a_p \times a_s) \cdot a_t}$	x		o	o			$\frac{\vec{p}_t}{3V}$	x		x	x		
$\frac{\vec{p}_p \times \vec{p}_s}{(a_p \times a_s) \cdot a_t}$		x	x		o	x	$-\frac{S_1}{3V} \vec{p}_t + \left(\frac{S_1}{3V} \times \frac{S_2}{3V}\right)$		x	x		x	x

Although in the case of QH modeling properties no. 4 and 5 are satisfied only for regular shapes, while in the case of TT modeling these properties are satisfied generally, in both cases (almost) regular shapes results in more efficient modeling. On the other side QH modeling based on (almost) regular shapes is much more efficient than corresponding TT modeling. Therefore, it can be concluded that QH combined with automatic mesh generation, that can provide high shape quality factor, is optimal.

Table 3. List of desired properties for surface and volume basis functions.

basis functions	1	2	3	4	5	6
surface	x		x	x		x
volume		x	x		x	x

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