

Conformal Galerkin testing for VIE using parametric geometry

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A moment method solution of a volume integral equation (VIE) using parametric geometry is presented. Typical Galerkin testing is shown not to be appropriate for curvilinear geometries, and a new testing scheme is proposed. By exploiting the orthogonality relationships between covariant and contravariant unitary vectors, testing functions in contravariant projection form and field expansion basis in covariant projection form were chosen.

Introduction: Volume integral equations (VIEs) are necessary for accurate electromagnetic modelling of three-dimensional high contrast inhomogeneous materials [1]. Recently, a VIE was derived to handle both dielectric and magnetic materials using a single field unknown [2]. This integral equation is quite attractive for composite materials involving non-trivial dielectric and magnetic properties and is considered here as our starting point.

Method of moments solutions of the VIE require the specifications of two sets of basis functions, namely the field expansion and testing functions. For parametric geometries, these vector functions consist of either contravariant vectors (covariant projection form) or covariant vectors (contravariant projection form). If edge-basis functions are considered over curvilinear hexahedral elements, then tangential field continuity can only be enforced if the field is represented in covariant projection form [3]. However, zero divergence can only be maintained for functions represented in contravariant projection form [4]. We found however that using Galerkin testing, where the testing functions are the same as the expansion functions, the covariant projection basis functions become inaccurate since the testing functions have nonzero divergence. In this Letter we propose a modified Galerkin implementation using contravariant vectors for the expansion functions and covariant vectors for the testing functions. In light of the defined inner product, the new testing scheme is shown to produce more accurate solutions and lower conditioned system matrices than previously reported.

Volume integral equation for parametric geometries: Consider the $N \times N$ moment method matrix system $[Z]\{\xi\} = \{b\}$ constructed via the weighted residual method. Specific matrix elements are given by [2]

$$Z_{j,i} = \langle \mathbf{t}_j(\mathbf{r}), \mathbf{e}_i(\mathbf{r}) \rangle - \left\langle \mathbf{t}_j(\mathbf{r}), k_0^2 \int_{V_i} d\mathbf{r}' \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot [\bar{\mathbf{e}}_i(\mathbf{r}') - \bar{\mathbf{I}}] \cdot \mathbf{e}_i(\mathbf{r}') \right\rangle + \left\langle \mathbf{t}_j(\mathbf{r}), \nabla \times \int_{V_i} d\mathbf{r}' \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot [\bar{\boldsymbol{\mu}}_r^{-1}(\mathbf{r}') - \bar{\mathbf{I}}] \cdot \nabla' \times \mathbf{e}_i(\mathbf{r}') \right\rangle \quad (1)$$

and the excitation column components are $b_j = \langle \mathbf{t}_j(\mathbf{r}), \mathbf{E}^{inc}(\mathbf{r}) \rangle$. Here, $\mathbf{e}_i(\mathbf{r})$ represent the basis functions, $\mathbf{t}_j(\mathbf{r})$ are the testing functions, $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ is the free space dyadic Green's function and $\langle \mathbf{f}, \mathbf{g} \rangle$ is the inner product defined as

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_V \mathbf{f}(\mathbf{r}) \cdot \mathbf{g}(\mathbf{r}) dV \quad (2)$$

Before discussing explicit forms of (1) using parametric geometry, we must introduce some notations and pertinent geometrical representations (for more details the reader is referred to [3–5]).

Similar to Euclidean geometries, parametric geometries require a set of co-ordinates to describe the characteristics of the vector space. One such set refers to the covariant unitary vectors given by $\mathbf{a}_i = \partial \mathbf{r} / \partial i$ for $i = u, v, w$. It is however important to note that for curved volumes and surfaces, these vectors do not form an orthogonal system. Conversely, the contravariant unitary vectors

$$\mathbf{a}^u = \frac{1}{V_+} (\mathbf{a}_v \times \mathbf{a}_w), \quad \mathbf{a}^v = \frac{1}{V_+} (\mathbf{a}_w \times \mathbf{a}_u), \quad \mathbf{a}^w = \frac{1}{V_+} (\mathbf{a}_u \times \mathbf{a}_v) \quad (3)$$

where $V_+ = \mathbf{a}_u \cdot (\mathbf{a}_v \times \mathbf{a}_w)$, do satisfy the desirable relation $\mathbf{a}^i \cdot \mathbf{a}_j = \delta_{ij}$, where δ_{ij} refers to the Kronecker delta for $i, j = u, v, w$. Because of this relation, it is readily seen that a vector field $\mathbf{E}(\mathbf{r})$ can be represented in

either covariant or contravariant projection form: $\mathbf{E}(\mathbf{r}) = (\mathbf{E} \cdot \mathbf{a}_u) \mathbf{a}^u + (\mathbf{E} \cdot \mathbf{a}_v) \mathbf{a}^v + (\mathbf{E} \cdot \mathbf{a}_w) \mathbf{a}^w$ or $\mathbf{E}(\mathbf{r}) = (\mathbf{E} \cdot \mathbf{a}^u) \mathbf{a}_u + (\mathbf{E} \cdot \mathbf{a}^v) \mathbf{a}_v + (\mathbf{E} \cdot \mathbf{a}^w) \mathbf{a}_w$. Fields expressed in the covariant projection form (with the proper choice of Nedelec basis functions) can explicitly satisfy tangential field continuity across faces of adjacent elements [6]. These are most commonly referred to as curl conforming expansions and are the natural choice for electromagnetic applications where the electric or magnetic intensity is unknown. If the fields are expressed in the contravariant projection form, normal field continuity can be enforced. These are appropriate for flux representations and are used to represent electric or magnetic current densities.

As in [4] our discretisation employs 27-point hexahedra the defining nodes of which are given by $\mathbf{r}_{ijk}$. Positions within the hexahedra can then be found from the vector $\mathbf{r}(u, v, w) = \sum_{i=0}^2 \sum_{j=0}^2 \sum_{k=0}^2 \mathbf{r}_{ijk} L_{ijk}(u, v, w)$, where the parametric co-ordinates $(u, v, w) \in [-1, 1], [-1, 1], [-1, 1]$ define a unit cube in (u, v, w) space and $L_{ijk}(u, v, w)$ are the appropriate Cartesian products of the quadratic Lagrange interpolation functions.

To evaluate $Z_{j,i}$ we must next proceed with the definition of the expansion basis and testing functions. For the hexahedral elements, we may choose either

$$\mathbf{E}_i(\mathbf{r}) = f_u^i(u, v, w) \mathbf{a}^u + f_v^i(u, v, w) \mathbf{a}^v + f_w^i(u, v, w) \mathbf{a}^w \quad (4)$$

or

$$\mathbf{E}_i(\mathbf{r}) = f_u^i(u, v, w) \frac{1}{V_i} \mathbf{a}_u + f_v^i(u, v, w) \frac{1}{V_i} \mathbf{a}_v + f_w^i(u, v, w) \frac{1}{V_i} \mathbf{a}_w \quad (5)$$

in which

$$f_u^i(u, v, w) = \zeta_1^{i,u}(1+v)(1+w) + \zeta_2^{i,u}(1+v)(1-w) + \zeta_3^{i,u}(1-v)(1+w) + \zeta_4^{i,u}(1-v)(1-w) \quad (6)$$

refers to the linear volumetric functions over the unit cube (f_v^i and f_w^i can be constructed from f_u^i by performing a cyclic rotation on $[u, v, w]$). We remark that, $\zeta_k^{i,u}$ is now associated with field unknowns, and represent the set of unknown coefficients to be found via the MoM. As stated earlier, (4) is a Nedelec curl conforming expansion, allowing for tangential field continuity to be enforced when proper pairing of the edge unknowns is performed. This expansion is also identical to that used by Antilla and Alexopoulos [3] in their finite element-boundary integral (FE-BI) formulation. Alternatively, form (5) has the property of zero-divergence within the volume elements, but is not divergence conforming in the Nedelec sense since normal field continuity cannot be enforced across the faces of the hexahedra when edge-basis functions are considered. Therefore, we will refer to these basis functions as pseudo-divergence conforming.

Conformal Galerkin testing: As mentioned earlier, typical Galerkin testing implies that the testing functions are equal to the field expansion functions. Therefore, if the curl basis functions are selected for the field expansion, they must be also chosen for testing. Doing so, and considering the scattering from a composite sphere of radius 0.2λ with $\epsilon_r = 2.2$ and $\mu_r = 1.37$, it is seen (see Fig. 1) that this choice produces rather inaccurate results for the RCS compared to a standard FE-BI solution that also uses the same hexahedral mesh and basis. Moreover, Fig. 2 shows that higher mesh densities will not generate a more accurate solution. Fig. 2 plots the average normal component error

$$\epsilon_{avg} = \sum_{i \in \text{faces}} \left(\int ds \frac{|E_{i,1}^\perp - E_{i,2}^\perp|}{\sqrt{|E_{i,1}|^2 + |E_{i,2}|^2}} \right) / N_{\text{faces}} \quad (7)$$

across volume faces, where N_{faces} is the total number of volume faces and the pair $(\mathbf{E}_{i,1}, \mathbf{E}_{i,2})$ are the calculated fields from the two respective volume elements sharing the i th volume face. This result points to a fundamental flaw in the used Galerkin formulation. Specifically, the error increases because the average error in the normal field component across the finite element faces increases with higher mesh density.

That is, although increasing the mesh density implies higher field resolution, it does come at the price of adding more testing functions with nonzero divergence which corrupts further the solution. However, if the testing functions are chosen to be the pseudo-divergence conforming basis in (5), errors in the solution are substantially reduced.

Fig. 1 confirms that this choice of testing produces the correct solution, and Fig. 2 demonstrates this fact by showing that the average error in the normal field component decreases with increasing mesh density.

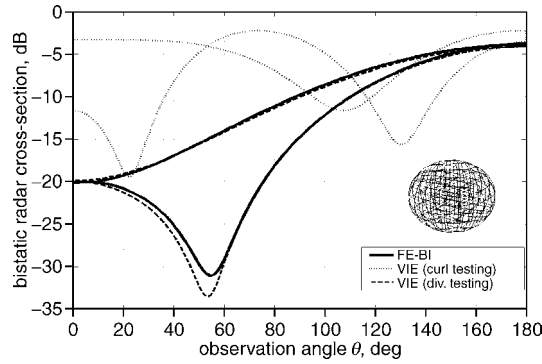


Fig. 1 Bistatic RCS for both pseudo-divergence and curl testing functions compared to FE-BI solution ($N = 300$)

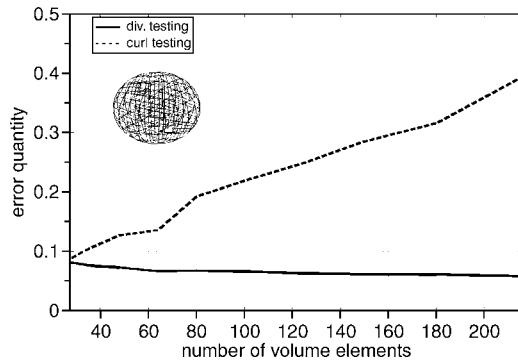


Fig. 2 Average error in normal field components across volume faces

We remark that for rectangular volumes both testing approaches lead to identical results and accuracy. Though this point is rather elementary, we must emphasise it to demonstrate that the proposed testing scheme is a form of Galerkin testing, only necessary for curvilinear geometries.

In light of the inner product definition (2), the new testing scheme exploits the Kronecker delta relationship between the covariant and contravariant unitary vectors. Thus, by considering inhomogeneous dielectric scatterers, we can compare our new testing scheme with a previously reported formulation [4] to confirm that our scheme is indeed optimal for curvilinear geometries. In [4], both the expansion and testing functions were chosen to be of the form in (5). However, in the newly proposed testing scheme, the orthonormality relationship between the covariant and contravariant vectors suggests a better conditioned matrix system should be produced. Additionally, the new testing scheme is expected to produce smaller solution errors for a given field resolution. To confirm our assumptions, we computed the

RCS error in Table 1 compared to the Mie series solution. Table 1 refers to a 'coated' sphere with an inner radius of 0.1λ ($\epsilon_{core} = 3.0 + i0.02$) and an outer radius of 0.2λ ($\epsilon_{coating} = 2.0 + i0.01$). As seen, the conformal Galerkin's testing scheme reduces both the error and the matrix condition number.

Table 1: Error and condition number for varying field resolution

Hex	Solution error		Matrix condition	
	Galerkin	Conformal Galerkin	Galerkin	Conformal Galerkin
32	0.0964	0.0755	74.8	57.3
81	0.0625	0.0449	138.8	60.4
160	0.0998	0.0500	260.3	68.8
256	0.0222	0.0194	94.0	52.5

Conclusion: A new conformal Galerkin's testing scheme is presented to solve volume integral equations via the method of moments when curvilinear finite elements are used for discretisation. With the proper choice of pseudo-divergence conforming testing functions and curl conforming basis functions, it is shown that the discretised system produced solutions that are more accurate and efficient than previously reported.

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